

# POSITIVE LYAPUNOV EXPONENTS FOR HAMILTONIAN LINEAR DIFFERENTIAL SYSTEMS

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**ABSTRACT.** We prove that for an open and dense set of Hölder symplectic cocycles over a non-uniformly hyperbolic diffeomorphism there are non-zero Lyapunov exponents with respect to any invariant ergodic measure with the local product structure. Moreover, we prove that there exists an open and dense set of Hamiltonian linear differential systems, over a suspension flow with bounded roof function, displaying at least one positive Lyapunov exponent. In consequence, typical cocycles over a uniformly hyperbolic flow are chaotic. Finally, we obtain similar results for cocycles over general flows preserving an ergodic, hyperbolic measure with local product structure.

## CONTENTS

1. Introduction	2
1.1. Symplectic cocycles and Hamiltonian linear differential systems	2
1.2. Lyapunov exponents	3
1.3. Overview	3
2. Statement of the main results	5
2.1. The discrete-time case	5
2.2. The continuous-time case	7
3. Preliminaries	11
3.1. The symplectic group of matrices	11
3.2. The symplectic geometry of Oseledets' spaces	12
3.3. Hyperbolic and suspension flows	14
3.4. Regularity and Lyapunov exponents of induced cocycles	15
4. Discrete-time symplectic cocycles	16
4.1. A quick tour on the proof of Theorem A	16
4.2. Zero Lyapunov exponents lead to rigidity	17
4.3. Obstructions using periodic points	19
4.4. Perturbation results	20
4.5. Finishing the proof of Theorem A	25
5. Hamiltonian linear differential systems over suspension flows	27
5.1. A reduction to the base dynamics	27
5.2. Continuous disintegration and criterion for non-zero Lyapunov exponents	29
5.3. Realization of symplectic time- $t$ actions by Hamiltonian linear differential systems	30

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5.4. Proof of Theorem B for suspension flows	34
6. Hamiltonian linear differential systems: General case	37
6.1. Non-uniform hyperbolicity for the flow and hyperbolicity for local Poincaré maps	37
6.2. Fiber-bunching for cocycles over Poincaré return maps	38
6.3. A lot of closed orbits inside holonomy blocks	40
6.4. Proof of Theorem B for general flows	42
Acknowledgements	43
References	43

## 1. INTRODUCTION

**1.1. Symplectic cocycles and Hamiltonian linear differential systems.** Let  $M$  denote a  $d$ -dimensional compact Hausdorff space  $M$ ,  $f: M \rightarrow M$  a diffeomorphism and  $A: M \rightarrow sp(2\ell, \mathbb{R})$  a continuous, sometimes smooth, map, where  $sp(2\ell, \mathbb{R})$ ,  $\ell \geq 1$ , stands for the symplectic Lie group of  $2\ell \times 2\ell$  matrices  $A$  and with entries over the reals such that  $A^T J A = J$ , where  $J$  is the skew-symmetric matrix defined on (3.2) and  $A^T$  is the transpose of  $A$ . The first part of this paper is to understand the dynamics defined by

$$A^n(x) = A(f^{n-1}(x)) \circ \dots \circ A(f(x)) \circ A(x),$$

for most systems  $A$  and when  $n \rightarrow \infty$ . This can be seen as a *toy model* aiming to understand the dynamics of symplectomorphisms, where  $A$  is defined to be the tangent map to a symplectomorphism  $f$  defined in a symplectic  $d$ -manifold with  $d = 2\ell$ .

In the second part we will study the time-continuous counterpart. Let  $X^t: M \rightarrow M$  be a continuous flow and  $H: M \rightarrow \mathfrak{sp}(2\ell, \mathbb{R})$  be a continuous, sometimes smooth, map, where  $\mathfrak{sp}(2\ell, \mathbb{R})$  denotes the Hamiltonian Lie algebra of  $2\ell \times 2\ell$  traceless matrices  $H$  and with entries over the reals such that  $JH + H^T J = 0$ . Given any  $x \in M$ , the solution  $u(t) = \Phi_H^t(x)$  of the non-autonomous linear differential equation  $\partial_t u(t) = H(X^t(\cdot)) \cdot u(t)$ , with initial condition  $\Phi_H^0(x) = \mathbf{1}_{2\ell}$ , where  $\mathbf{1}_{2\ell}$  is the identity matrix in  $\mathbb{R}^{2\ell}$ , is a linear flow which evolves in the symplectic linear group  $sp(2\ell, \mathbb{R})$ . The transversal linear Poincaré flow (see [8, §2.3]) of a Hamiltonian flow defined in a  $2\ell$ -dimensional manifold and such that  $\|\partial_t X^t(x)|_{t=0}\| = \|X(x)\| \neq 0$  for all regular points  $x$  is the common example of a non-autonomous Hamiltonian linear differential system.

The framework of symplectic cocycles (respectively, Hamiltonian linear differential systems) is a good start if one aims to understand the behavior of the dynamical cocycle associated to a given symplectomorphism (respectively, Hamiltonian flow). However, the cocycle or linear differential systems *Achilles heel* tends to be the independent relation between the base and fiber dynamics which is a natural counterweight to its great generality. In other words we are able to perturb the fiber keeping unchanged the base dynamical system (or vice-versa) but on the other hand we allow a vast number of symplectic actions in the fiber. We should keep in mind that any perturbation in the action of the fiber of a dynamical cocycle should begin with a perturbation in the dynamical system itself which, in general, cause extra difficulties.

**1.2. Lyapunov exponents.** Given a linear differential system  $H$  over a flow  $X^t$  the Lyapunov exponents detect if there are any exponential asymptotic behavior on the evolution of the time-continuous cocycle  $\Phi_H^t$  along orbits (cf. [5]). If the flow is over a fixed point then  $H(t) = H$  is constant, hence the Lyapunov exponents are exactly the real parts of the eigenvalues of  $H$ . In general, the eigenvalues of the matrix  $H(t)$  are meaningless if one aims to study the asymptotic solutions. Under certain measure preserving assumptions on  $X^t$  and smoothness of  $H$  the existence of Lyapunov exponents for almost every point is guaranteed by Oseledets' theorem ([32, 24]). Non-zero Lyapunov exponents assure, in average, exponential rate of divergence or convergence of two neighboring trajectories, whereas zero exponents give us the lack of any kind of average exponential behavior. A flow is said to be *nonuniformly hyperbolic* if its Lyapunov exponents are all different from zero. The correspondent definitions for the discrete-time case are completely analogous.

A central question in dynamical systems is to determine whether we have non-zero Lyapunov exponents for the original dynamics and some or the majority of nearby systems. Such an answer usually depends on the smoothness and richness of the dynamical system, among other aspects.

**1.3. Overview.** Concerning with continuous flows over compact Hausdorff spaces, and motivated by the works by Bochi and Viana [11, 13] for discrete dynamical systems, the first author proved in [6, 7] that there exists a residual  $\mathcal{R}$ , i.e. a  $C^0$ -dense  $G_\delta$ , such that any conservative linear differential system in  $\mathcal{R}$  yields the dichotomy: either the Oseledets decomposition along the orbit of almost every point has a weak form of hyperbolicity called *dominated splitting* or else the spectrum is trivial mean that all the Lyapunov exponents vanish. The main idea behind the proof of these results, also used by Novikov [31] and by Mañé [27], is to use the absence of dominated splitting to cause a decay of the Lyapunov exponents by perturbing the system rotating Oseledets' subspaces thus mixing different expansion rates.

Still in the  $C^0$ -topology, Millionshchikov [28, 29] exhibited a  $C^0$  open and dense set in a certain class of linear differential systems displaying a simple spectrum, that is, the Lyapunov exponents are all different. Moreover, Fabbri [22] proved the  $C^0$ -open and denseness of hyperbolicity on the torus for 2-dimensional linear differential systems. Finally, Nerurkar [30] proved the positivity of Lyapunov exponents for a dense set in a class of conservative linear differential systems.

Let us mention that, in [21], Cong proved that a discrete generic bounded cocycle has simple spectrum and that the Oseledets splitting is dominated. In consequence, uniform hyperbolicity is generic among discrete area-preserving *bounded* cocycles.

Other approaches were given in [2, 3, 10] where it was proved abundance of trivial spectrum but with respect to  $L^p$  topologies. Let us also mention [12] where the authors, instead of perturbing the fiber, perturb the base dynamics and recover again the dichotomy: hyperbolicity *versus* zero Lyapunov exponents.

In this work we are interested in proving abundance of non-zero Lyapunov exponents. In fact, a major breakthrough in the analysis of the Lyapunov exponents of Hölder continuous cocycles over non-uniformly hyperbolic base map was obtained recently by an outstanding paper by Viana [38] and our purpose here is to contribute to the better understanding of the ergodic

theory of symplectic cocycles and Hamiltonian linear differential systems and to answer some of the questions raised in that article, namely Problem 4 and part of Problem 6 in [38, pp. 678]. More precisely, we generalize to the setting of symplectic cocycles and of Hamiltonian linear differential systems the results obtained by Viana [38] for conservative cocycles. Our approach will be divided into two parts mainly because the extension of many results concerning discrete dynamical systems to the time-continuous setting is usually far from being immediate and there is no direct approach to translate results from both settings as we will now discuss. First, it is proven that fiber-bunched cocycles admit center dynamics called holonomies. Then, using a generalization of Ledrappier's criterium ([26]), Viana proved that zero Lyapunov exponents correspond to a highly non-generic condition on the system  $A$ : conditional measures associated to invariant measures for the cocycle are preserved under holonomies. Finally, for  $sl(d, \mathbb{R})$ -cocycles the map  $A \mapsto H_{A,x,y}$  is a submersion and this leads to show that the set of cocycles  $A \in C^{r,\nu}(M, sl(d, \mathbb{R}))$  satisfying the later is a closed subset of empty interior. The case of  $sp(2\ell, \mathbb{R})$ -symplectic cocycles have more subtleties as pointed out by Viana [38, page 678], since the fundamental and elegant lemma asserting that the holonomy maps are submersions as function of  $A \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  fails to be true because the symplectic group  $sp(2\ell, \mathbb{R})$  has dimension  $\ell(2\ell + 1)$  which is smaller than the necessary dimension  $2\ell(2\ell - 1)$ , ( $\ell \geq 2$ ). This lead to the question of understanding which groups can be taken to obtain non-trivial spectrum. In this article we use a symplectic perturbative approach (see §4.4) in small neighborhoods of heteroclinic points to show that every cocycle  $A$  is  $C^{r+\nu}$ -approximated by open sets of cocycles so that unstable holonomies remain unchanged while stable holonomies are modified in order *not* to satisfy a rigid condition. The strategy used to prove the result for Hamiltonian linear differential systems over suspension flows is to make a reduction to the discrete time case. For that we will consider an induced cocycle in the fiber that also depends on the roof function. It is here that we need to require the roof function to be bounded. At this point one could expect the result to yield immediately in our discrete-time result but note that this is not the case. Actually, in spite of perturbing the discrete time cocycles  $B \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  in order to make the cocycle in some sense typical situation, our perturbations (see §5.3) are on the space of its “infinitesimal generators” or, more accurately, on the Hamiltonian linear differential system  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{R}))$  generating a fundamental solution  $\Phi'_H$  over the suspension flow  $(X^t)_t$ . One of the main difficulties is really to analyze the variation of the holonomies for the reduced cocycle  $\Psi_H$  as a function of the infinitesimal generators  $H$ . On the other hand, one could hope that it is possible to obtain a proof of the discrete corresponding theorems for flows by reducing to the time-one diffeomorphism  $f = X^1$ . This is known as the *embedding problem*: given a diffeomorphism can it be embedded as the time-one map of a flow? However, it is well-known that  $C^1$  diffeomorphisms that embed in  $C^1$  flows form a nowhere dense set (see [33]).

The general case of flows, not necessarily suspensions, is a little more complicated. Actually, we do not have the whole hyperbolic discrete-time structure given by the suspension map and thus, we must reformulate it for continuous-time systems. As usually, the study of the hyperbolic map projected in the normal bundle and also the analysis of induced return maps is of great help.

Our paper is organized as follows. In Section 2 we state our main results. We collect some preliminary results on groups of symplectic matrices, symplectic geometry of Oseledets spaces, hyperbolic and suspension flows and Pesin theory and Lyapunov exponents of induced cocycles in Section 3, while the proofs of the main results are given in Sections 4, 5 and 6. In Section 4 we deal with the symplectic cocycles case, in Section 5 we treat the Hamiltonian linear differential systems in the particular case when the base dynamics evolve in a suspension flow and, finally, in Section 6 we obtain the full statement for Hamiltonian linear differential systems over general flows.

## 2. STATEMENT OF THE MAIN RESULTS

**2.1. The discrete-time case.** This section is devoted to recall some necessary notions and to state our main results. Our first result answers in an affirmative way to Problem 4 in [38]. We recall some definitions. Given  $A \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  we will denote, by a slight abuse of notation, by *cocycle* the skew-product

$$F_A: \begin{array}{ccc} M \times \mathbb{K}^{2\ell} & \longrightarrow & M \times \mathbb{K}^{2\ell} \\ (x, v) & \longrightarrow & (f(x), A(x)v) \end{array}$$

where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . We denote by  $f_A$  the natural cocycle induced by  $F_A$  in the projective spaces  $P\mathbb{K}^{2\ell}$ . If  $\mu$  is an  $f$ -invariant probability measure such that  $\log \|A^{\pm 1}\| \in L^1(\mu)$  then it follows from Oseledets theorem ([32]) that for  $\mu$ -almost every  $x$  there exists a decomposition  $\mathbb{K}^d = E_x^1 \oplus E_x^2 \oplus \dots \oplus E_x^{k(x)}$ , called, the *Oseledets splitting*, and for  $1 \leq i \leq k(x)$  there are well defined real numbers

$$\lambda_i(A, f, x) = \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|A^n(x)v_i\|, \quad \forall v_i \in E_x^i \setminus \{\vec{0}\}$$

called the *Lyapunov exponents* associated to  $A$ ,  $f$  and  $x$ . It is well known that, if  $\mu$  is ergodic, then the Lyapunov exponents are almost everywhere constant. Since we are dealing with symplectic cocycles and  $sp(2\ell, \mathbb{R}) \subset sl(2\ell, \mathbb{R})$ , this implies that  $\sum_{i=1}^{k(x)} \lambda_i(A, f, x) = 0$ . Notice that the spectrum of a symplectic linear transformation is symmetric with respect to the  $x$ -axis and to  $\mathbb{S}^1$ . In fact, if  $\sigma \in \mathbb{C}$  is an eigenvalue with multiplicity  $m$  so is  $\sigma^{-1}$ ,  $\bar{\sigma}$  and  $\bar{\sigma}^{-1}$  keeping the same multiplicity (see e.g. [35, Proposition 1.5]). Therefore, since Lyapunov exponents come in pairs in the symplectic setting, then  $\lambda_i(A, f, x) = -\lambda_{2\ell-i+1}(A, f, x) := -\lambda_i(A, f, x)$  for all  $i \in \{1, \dots, \ell\}$ . So, not counting the multiplicity and abbreviating  $\lambda(A, f, x) = \lambda(x)$ , we have the increasing set of real numbers,

$$\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_\ell(x) \geq 0 \geq -\lambda_\ell(x) \geq \dots \geq -\lambda_2(x) \geq -\lambda_1(x),$$

or, equivalently,

$$\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_\ell(x) \geq 0 \geq \lambda_\ell(x) \geq \dots \geq \lambda_2(x) \geq \lambda_1(x).$$

Associated to the Lyapunov exponents we have the Oseledets splitting

$$\mathbb{K}^{2\ell} = E_x^1 \oplus E_x^2 \oplus \dots \oplus E_x^\ell \oplus E_x^{\hat{\ell}} \oplus \dots \oplus E_x^{\hat{2}} \oplus E_x^{\hat{1}}. \quad (2.1)$$

We will see in §3.1 that the vector space  $\mathbb{K}^{2\ell}$  can be decomposed into  $\ell$  two-dimensional symplectic subspaces.

Recall that an  $f$ -invariant probability measure  $\mu$  is *hyperbolic* if it has only non-zero Lyapunov exponents. In that case, for every regular point  $x$  let  $E_x^s$  (respectively  $E_x^u$ ) denote the sums of all Lyapunov subspaces corresponding to all negative (respectively positive) Lyapunov exponents. It follows from Pesin's stable manifold theorem (see e.g. [5]) that for  $\mu$ -almost every  $x$  there exists a  $C^1$ -embedded disk  $W_{\text{loc}}^s(x)$  (local stable manifold at  $x$ ) such that  $T_x W_{\text{loc}}^s(x) = E_x^s$ , it is forward invariant  $f(W_{\text{loc}}^s(x)) \subset W_{\text{loc}}^s(x)$  and the following holds: given  $0 < \tau_x < \min\{|\lambda_i(x)| : \lambda_i(x) < 0\}$  there exists a positive constant  $K_x$  such that  $d(f^n(y), f^n(z)) \leq K_x e^{-n\tau_x} d(y, z)$  for every  $y, z \in W_{\text{loc}}^s(x)$ . Local unstable manifolds  $W_{\text{loc}}^u(x)$  are defined analogously using  $E_x^u$  and  $f^{-1}$ .

Moreover, since local invariant manifolds and the constants above vary measurably with the point  $x$  one can consider *hyperbolic blocks*  $\mathcal{H}(K, \tau)$  (sometimes with the notation  $\Lambda_{\tau, K}$  used also in [25, 38]) with measure arbitrary close to 1 by taking larger  $K$  and smaller  $\tau$  in such a way that  $K_x \leq K$ ,  $\tau_x \geq \tau$  and both the local invariant manifolds  $W_{\text{loc}}^s(x)$  and  $W_{\text{loc}}^u(x)$  vary continuously with  $x \in \mathcal{H}(K, \tau)$ . In consequence, if  $x \in \mathcal{H}(K, \tau)$  and  $\delta > 0$  is small enough, then for every  $y, z \in B(x, \delta)$  the intersection  $[y, z] := W_{\text{loc}}^u(y) \cap W_{\text{loc}}^s(z) \neq \emptyset$  consists of a unique point. Set  $\mathcal{N}_x^u(\delta) = \{[x, y] \in W_{\text{loc}}^u(x) : y \in \mathcal{H}(K, \tau) \cap B(x, \delta)\}$  to be a  $u$ -neighborhood of  $x$  and  $\mathcal{N}_x^s(\delta) = \{[y, x] \in W_{\text{loc}}^s(x) : y \in \mathcal{H}(K, \tau) \cap B(x, \delta)\}$  an  $s$ -neighborhood of  $x$ . It is not hard to check that the map

$$\begin{aligned} \Upsilon_x : \mathcal{N}_\delta(x) &\rightarrow \mathcal{N}_x^u(\delta) \times \mathcal{N}_x^s(\delta) \\ y &\mapsto ([x, y], [y, x]) \end{aligned}$$

is a homeomorphism, where  $\mathcal{N}_\delta(x) := \mathcal{H}(K, \tau) \cap B(x, \delta)$  is a neighborhood of  $x$  in  $\mathcal{H}(K, \tau)$ . Now, we recall the notion of local product structure (cf. [38, Page 646]).

**Definition 2.1.** A hyperbolic measure  $\mu$  has *local product structure* if for every  $x \in \text{supp}(\mu)$  ( $\text{supp}(\mu)$  stands for the support of the measure  $\mu$ ) and a small  $\delta > 0$  the measure  $\mu|_{\mathcal{N}_x(\delta)}$  is equivalent to the product measure  $\mu_x^u \times \mu_x^s$ , where  $\mu_x^i$  denotes the conditional measure of  $(\Upsilon_x)_*(\mu|_{\mathcal{N}_x(\delta)})$  on  $\mathcal{N}_x^i(\delta)$ , for  $i \in u, s$ .

At this point we describe the space of cocycles that we shall consider. We say that  $A : M \rightarrow sp(2\ell, \mathbb{K})$  is a  $C^{r, \nu}$ -cocycle if the map  $A$  is  $r$ -times differentiable and  $D^r A$  is  $\nu$ -Hölder continuous and denote by  $C^{r, \nu}(M, sp(2\ell, \mathbb{K}))$  the vector space of  $C^{r, \nu}$ -cocycles. It is not difficult to prove that the vector space  $C^{r, \nu}(M, sp(2\ell, \mathbb{K}))$  of  $C^{r, \nu}$ -cocycles endowed with the norm  $\|\cdot\|_{r, \nu}$  defined as

$$\|A\|_{r, \nu} = \sup_{0 \leq j \leq r} \|D^j A(x)\| + \sup_{x \neq y} \frac{\|D^r A(x) - D^r A(y)\|}{d(x, y)^\nu},$$

is a Banach space. Let us also mention that for the proofs it is enough to consider the case when  $\nu = 1$ , that is, of Lipschitz matrices. In fact, if  $A$  is  $\nu$ -Hölder continuous with respect to the metric  $d(\cdot, \cdot)$  then it is Lipschitz with respect to the metric  $d(\cdot, \cdot)^\nu$ . Hence, up to a change of metric we may assume that  $A$  is Lipschitz and we will do so throughout the paper. We are now in a position to state our first main result.

**Theorem A.** *Let  $M$  be a compact Riemannian manifold. Take  $f \in \text{Diff}^{1+\alpha}(M)$  ( $\alpha > 0$ ) and an  $f$ -invariant, ergodic and hyperbolic probability measure  $\mu$  with local product structure. Then,*

there exists an open and dense set of maps  $\mathcal{O}$  in  $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  such that for any  $A \in \mathcal{O}$  the cocycle  $F_A$  has at least one positive Lyapunov exponent at  $\mu$ -almost every point. Moreover, the complement is a set with infinite codimension.

It follows from the symplectic geometry of the Oseledets subspace (see §3.2) that previous result also assures that the cocycle  $F_A$  has at least one negative Lyapunov exponent at  $\mu$ -almost every point.

Let us recall the notion of fiber-bunched (or dominated) cocycles over some uniformly hyperbolic homeomorphisms. Let  $X$  be a compact metric space, let  $f : X \rightarrow X$  be an homeomorphism and  $\Lambda \subset X$  be a compact  $f$ -invariant set. We say that  $f|_\Lambda$  is *uniformly hyperbolic* if there exists a distance  $d$  and constants  $0 < \lambda < 1$  and  $\varepsilon, \delta > 0$  such that:

- (i)  $d(f^n(y), f^n(z)) \leq \lambda^n d(y, z)$  for all  $y, z \in W_\varepsilon^s(x)$  and  $n \geq 0$ ;
- (ii)  $d(f^{-n}(y), f^{-n}(z)) \leq \lambda^n d(y, z)$  for all  $y, z \in W_\varepsilon^u(x)$  and  $n \geq 0$ ;
- (iii) if  $x, y \in \Lambda$  and  $d(x, y) < \delta$  then the intersection  $[x, y] := W_\varepsilon^u(x) \cap W_\varepsilon^s(y)$  consists of exactly one point and depends continuously on  $x$  and  $y$ .

For simplicity we say that  $\Lambda$  is a uniformly hyperbolic set.

**Definition 2.2.** Let  $X$  be a compact metric space,  $f : X \rightarrow X$  be a homeomorphism and  $\Lambda \subset X$  is a uniformly hyperbolic set. We say that an  $\alpha$ -Hölder continuous cocycle  $A : \Lambda \rightarrow sp(2\ell, \mathbb{K})$  is *fiber-bunched* if  $\|A(x)\| \|A(x)^{-1}\| \lambda^\alpha < 1$  for all  $x \in \Lambda$ .

In view of the recent developments by Avila and Viana [4] we expect that most of the discrete time and time-continuous fiber-bunched symplectic cocycles do have simple Lyapunov spectrum. We make these assertions more precisely:

**Conjecture 1:** If  $(f, \mu)$  is as above then there exists a residual subset of fiber-bunched maps  $\mathcal{R} \subset C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  such that each cocycle  $F_A$  has simple spectrum for every  $A \in \mathcal{R}$ .

We note that this conjecture holds true in the class of *fiber-bunched* cocycles and the base transformation has an at most countable Markov partition. This is a consequence of the work of [4], using that the *twisting* and *pinching* conditions extend to the symplectic setting. Moreover, the following conjecture, which is the symplectic version of Bonatti, Viana [16] remains open:

**Conjecture 2:** Let  $\Lambda$  be a hyperbolic set and  $\mu$  be an  $f$ -invariant probability measure with local product structure. Then there exists a residual subset  $\mathcal{R} \subset C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  of *fiber-bunched* maps such that each cocycle  $F_A$  has simple spectrum for every  $A \in \mathcal{R}$ .

More recently, Cambrinha [19] has announced an affirmative solution to the previous conjectures. The time-continuous versions still remain open questions. We proceed to deal with time-continuous symplectic cocycles.

**2.2. The continuous-time case.** Our next results deals with time-continuous Hamiltonian linear differential systems which are also called Hamiltonian skew-product flows. Let  $sp(2\ell, \mathbb{K})$  denote the symplectic Lie algebra and let  $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  denote the Banach space of  $C^{r+\nu}$  linear differential systems with values on the Lie algebra  $sp(2\ell, \mathbb{K})$ . Given  $H \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$

and a smooth flow  $X^t: M \rightarrow M$ , for each  $x \in M$  we consider the non-autonomous linear differential equation

$$\partial_t u(s) \Big|_{s=t} = H(X^t(x)) \cdot u(t), \quad (2.2)$$

known as *linear variational equation* (or equation of first variation). Fixing the initial condition  $u(0) = \mathbf{1}_{2\ell}$  the unique solution of (2.2) is called the *fundamental solution* (or *fundamental matrix*) related to the system  $H$ . The solution of (2.2) is a linear flow  $\Phi_H^t(x): \mathbb{K}_x^d \rightarrow \mathbb{K}_{X^t(x)}^d$  which may be seen as the skew-product flow

$$\begin{aligned} \Phi_H^t: \quad M \times \mathbb{K}^d &\longrightarrow M \times \mathbb{K}^d \\ (x, v) &\longrightarrow (X^t(x), \Phi_H^t(x)v). \end{aligned}$$

Moreover, the cocycle identity  $\Phi_H^{t+s}(x) = \Phi_H^s(X^t(x)) \circ \Phi_H^t(x)$  holds for all  $x \in M$  and  $t, s \in \mathbb{R}$ . Furthermore, the transformation  $H$  satisfies  $H(x) = \partial_t \Phi_H^t(x)|_{t=0}$  for all  $x \in M$  and it is referred as the *infinitesimal generator* associated to  $\Phi_H^t$ . It follows from the previous cocycle identity that, for every  $x \in M$  and  $t \in \mathbb{R}$ ,

$$(\Phi_H^t(x))^{-1} = \Phi_H^{-t}(X^t(x)). \quad (2.3)$$

This coincides with the solution of the differential equation associated to the infinitesimal generator  $-H$ , that is,  $\partial_t u(s)|_{s=t} = -H(X^t(x)) \cdot u(t)$ , because time is reversed. The Lyapunov exponents of  $\Phi_H^t$  are defined analogously to the discrete time setting.

Let us consider a nonuniformly hyperbolic flow  $X^t: M \rightarrow M$ , that is, a flow such that all the Lyapunov exponents of its dynamical cocycle  $DX^t$  are different from zero (except of course the flow direction which has always zero Lyapunov exponent). Assume that there exists a set  $\mathbf{N} \subset M$  with negative Lyapunov exponents for  $X$ . For a full detailed exposition about the formalism of Lyapunov exponents for flows see [9]. For any  $x \in \mathbf{N}$  we fix  $\tau_x$  such that  $0 < \tau_x < \min\{|\lambda_i(x)| : \lambda_i(x) < 0\}$ . There exist a measurable function  $K: \mathbf{N} \rightarrow (0, +\infty)$  and, given any  $x \in \mathbf{N}$ , a *local stable manifold*  $W_{loc}^s(x)$  such that:

- (i)  $T_x W_{loc}^s(x) = E_x^s$ , i.e. the stable subspace integrates on  $W_{loc}^s(x)$  and
- (ii)  $d(X^t(x), X^t(y)) \leq K_x e^{-\tau_x t} d(x, y)$ , for every  $y \in W_{loc}^s(x)$ .

Let  $W^s(x) := \{X^{-t}(W_{loc}^s(X^t(x))); t > 0\}$  stands for the *global stable manifold* of  $x$ . The saturate of  $W^s(x)$  defined by  $\cup_{t \in \mathbb{R}} X^t(W^s(x)) = \cup_{t \in \mathbb{R}} W^s(X^t(x))$  and denoted by  $W^{ws}(x)$  is sometimes called *weak stable manifold*. In an analogous way, and of course if the flow has positive Lyapunov exponents, we define *local unstable*, *unstable* and *weak stable manifolds* of  $x$ , which are the stable objects for the flow  $X^{-t}$  and are denoted respectively by  $W_{loc}^u(x)$ ,  $W^u(x)$  and  $W^{wu}(x)$ . Like in diffeomorphisms these objects vary measurably with the point  $x$  and so one can consider *hyperbolic blocks*  $\mathcal{H}(K, \tau)$  on which both the local invariant manifolds  $W_{loc}^s(x)$  and  $W_{loc}^u(x)$  are uniform and vary continuously with  $x \in \mathcal{H}(K, \tau)$ .

Oftentimes it is useful to work with the *linear Poincaré flow* of the vector field  $X$ . This linear flow is defined by the projection on the normal subsection of the flow  $N$ , formally by  $P_X^t(p) := \Pi_{X^t(p)} \circ DX_p^t$  where  $\Pi_{X^t(p)}$  stands for the projection in the normal fiber  $N_{X^t(p)}$  at  $X^t(p)$ . In fact the linear Poincaré flow is the derivative of the Poincaré map of the flow  $\mathcal{P}_X^t$ .

The following result says that, if  $X$  is nonuniformly hyperbolic, then so it is its linear Poincaré flow.



**Lemma 2.3.** *Let  $X^t$  be a nonuniformly hyperbolic flow associated to the measure  $\mu$  with invariant decomposition  $E_1^s(x) \oplus \dots \oplus E_i^s(x) \oplus E_1^u(x) \oplus \dots \oplus E_j^u(x)$  on a  $\mu$ -generic point  $x$  and with associated Lyapunov exponents respectively  $\lambda_1^s(x), \dots, \lambda_i^s(x) < 0$  and  $\lambda_1^u(x), \dots, \lambda_j^u(x) > 0$ . The Lyapunov exponents of  $P_X^t(x)$  associated to the projected decomposition  $N_1^s(x) \oplus \dots \oplus N_i^s(x) \oplus N_1^u(x) \oplus \dots \oplus N_j^u(x)$ , where  $N_k^\sigma(x) = \Pi_x(E_k^\sigma(x))$ , are respectively  $\lambda_1^s(x), \dots, \lambda_i^s(x) < 0$  and  $\lambda_1^u(x), \dots, \lambda_j^u(x) > 0$ .*

*Proof.* Let us consider any  $\mu$ -generic point  $x$  and  $u \in N_k^u(x) \setminus \{\vec{0}\}$ , for some  $k = 1, \dots, j$ , and denote by  $\theta_t(x) = \angle(X(X^t(x)), E^u(X^t(x)))$  the angle between the flow direction  $X(X^t(x))$  and the direction  $E^u(X^t(x))$ . Then, for some  $\alpha \in \mathbb{R}$  and  $v \in E_k^u(x)$ , we have

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|P_X^t(x) \cdot u\| &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\Pi_{X^t(x)} \circ DX_x^t \cdot (\alpha X(x) + v)\| \\ &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\alpha \Pi_{X^t(x)} \circ DX_x^t(X(x)) + \Pi_{X^t(x)} \circ DX_x^t \cdot v\| \\ &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\alpha \Pi_{X^t(x)} \circ X(X^t(x)) + \Pi_{X^t(x)} \circ DX_x^t \cdot v\| \\ &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log (\sin(\theta_t(x)) \|DX_x^t \cdot v\|) \\ &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sin(\theta_t(x)) + \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|DX_x^t \cdot v\| \\ &= \lambda_k^u(x), \end{aligned}$$

because  $\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sin(\theta_t(x)) = 0$  which is a consequence of the sub exponential growth of the angle given by the Oseledets theorem (see e.g. [5]). The calculus for  $N_k^s(x)$ ,  $k = 1, \dots, i$ , is analog.  $\square$

It follows from Pesin's theory (see [5]) and Lemma 2.3 that given  $x \in \mathbf{N}$ , where  $\mathbf{N} \subset M$  is a set with negative Lyapunov exponents for  $X$ , thus with negative Lyapunov exponents for the Poincaré map  $\mathcal{P}_X^t$ , and fixed  $\tau_x$  such that  $0 < \tau_x < \min\{|\lambda_i(x)| : \lambda_i(x) < 0\}$ , there exist a measurable function  $\tilde{K} : \mathbf{N} \rightarrow (0, +\infty)$  and a *local stable normal manifold*  $\mathcal{N}_{loc}^s(x)$  such that:

- (i)  $T_x \mathcal{N}_{loc}^s(x) = N^s(x)$  and
- (ii)  $d(\mathcal{P}_X^t(x), \mathcal{P}_X^t(y)) \leq \tilde{K}_x e^{-\tau_x t} d(x, y)$ , for every  $y \in \mathcal{N}_{loc}^s(x)$ .

We observe that, in local charts on  $\mathbb{R}^d$ ,  $\mathcal{N}_{loc}^s(x)$  is the intersection of  $W^{ws}(x)$  with a local small normal section to the flow on  $x$ .

Now we proceed to describe the local product structure for non-atomic  $(X^t)_t$ -invariant and ergodic probability measures  $\mu$  since it will be clear for atomic measures. For that we will recall the existence of tubular neighborhoods (see e.g. [1, 34]). Given a regular point  $x \in M$  for the vector field  $X$  the *tubular neighborhood theorem* asserts the existence of a positive  $\delta = \delta_x > 0$ , an open neighborhood  $U_x^\delta$  of  $x$ , and a diffeomorphism  $\Psi_x : U_x^\delta \rightarrow (-\delta, \delta) \times B(x, \delta) \subset \mathbb{R} \times \mathbb{R}^d$ ,  $B(x, \delta)$  is identified with the ball  $B(\vec{0}, \delta) \cap \langle (1, 0, \dots, 0)^\perp \rangle$ , where  $\langle (1, 0, \dots, 0)^\perp \rangle$  stands for the hyperspace perpendicular to the vector  $(1, 0, \dots, 0)$ , such that the vector field  $X$  on  $U_x^\delta$  is the pull-back of the vector field  $Y := (1, 0, \dots, 0)$  on  $(-\delta, \delta) \times B(x, \delta)$ . More precisely,

$Y = (\Psi_x)_* X := D(\Psi_x)_{\Psi_x^{-1}} X(\Psi_x^{-1})$ . In this case the associated flows are conjugated, that is,  $Y^t(\cdot) = \Psi_x(X^t(\Psi_x^{-1}(\cdot)))$  for  $t$  small enough.

So, given  $x \in \mathcal{H}(K, \tau)$  and  $\varepsilon > 0$  small enough the size of both invariant manifolds  $W_{loc}^u(X^t(y))$  and  $W_{loc}^s(X^t(y))$  have size at least  $\varepsilon$  for all  $y \in \mathcal{H}(K, \tau) \cap U_x^\delta$  and all  $t$  so that  $X^t(y) \in U_x^\delta$ . In consequence, if one considers the section  $\Sigma_x = \Psi_x^{-1}(\{0\} \times B(x, \delta))$  through the point  $x$  then for any  $y \in \mathcal{H}(K, \tau) \cap U_x^\delta$  the intersection  $\mathcal{F}_y^s = W_{loc}^{ws}(y) \cap \Sigma_x$  (respectively  $\mathcal{F}_y^u = W_{loc}^{wu}(y) \cap \Sigma_x$ ) defines a smooth and long stable (respectively unstable) leaf on  $\Sigma_x$ .

Since the angles of strong stable and unstable foliations are bounded away from zero on hyperbolic blocks it is not hard to check that for all  $y, z \in \mathcal{H}(K, \tau) \cap U_x^\delta$  the intersection  $[y, z]_{\Sigma_x} := \mathcal{F}_y^u \cap \mathcal{F}_z^s$  consists of a unique point, provided that  $\delta$  is small.

Set

$$\mathcal{N}_x^u(\delta) = \{[x, y]_{\Sigma_x} \in \mathcal{F}_x^u : y \in \mathcal{H}(K, \tau) \cap U_x^\delta\} \subset \Sigma_x$$

to be a  $u$ -neighborhood of  $x$  in  $\Sigma_x$  and

$$\mathcal{N}_x^s(\delta) = \{[y, x]_{\Sigma_x} \in \mathcal{F}_x^s : y \in \mathcal{H}(K, \tau) \cap U_x^\delta\} \subset \Sigma_x$$

an  $s$ -neighborhood of  $x$  in  $\Sigma_x$ .

Set  $\mathcal{N}_\delta(x) := \mathcal{H}(K, \tau) \cap U_x^\delta$  to be a neighborhood of  $x$  in  $\mathcal{H}(K, \tau)$ . Then the map

$$\begin{aligned} \Upsilon_x : \mathcal{N}_\delta(x) &\longrightarrow \mathcal{N}_x^u(\delta) \times \mathcal{N}_x^s(\delta) \times (-\delta, \delta) \\ y &\longrightarrow ([x, y]_{\Sigma}, [y, x]_{\Sigma}, t(y)), \end{aligned}$$

with  $X^{t(y)}(y) \in \Sigma_x$  is a homeomorphism. Hence, we can now define local product structure for flows.

**Definition 2.4.** A hyperbolic measure  $\mu$  has *local product structure* if for every  $x \in \text{supp}(\mu)$  and a small  $\delta > 0$  the measure  $\mu|_{\mathcal{N}_x(\delta)}$  is equivalent to the product measure  $\mu_x^u \times \mu_x^s \times \text{Leb}$ , where  $\mu_x^i$  denotes the conditional measure of  $(\Upsilon_x)_*(\mu|_{\mathcal{N}_x(\delta)})$  on  $\mathcal{N}_x^i(\delta)$ , for  $i \in u, s$  and  $\text{Leb}$  stands for the 1-dimensional Lebesgue measure along the flow direction. We denote by  $\mu_\Sigma$  the marginal measure of  $\mu|_{\mathcal{N}_x(\delta)}$  on  $\Sigma$  obtained via projection along the flow direction.

**Remark 2.5.** Our previous computations show that hyperbolic blocks for flows locally contain pieces of orbits. Moreover, it will be important to notice that the previous property implies that, up to reduce  $B(x, \varepsilon)$ , it will be independent of the size of the tube by the long tubular neighborhood theorem.

**Remark 2.6.** Our definition does not depend on the smooth cross section through the point. In particular we may assume without loss of generality that local strong stable and unstable manifolds through  $x$  are contained in  $\Sigma_x$  and consequently  $\mathcal{F}_x^s \subset W_{loc}^s(x)$  and  $\mathcal{F}_x^u \subset W_{loc}^u(x)$ .

Let us mention that despite the fact that it seems a strong condition, the local product structure property is satisfied by a broad class of examples as all equilibrium states associated to Hölder continuous potentials and Axiom A flows (see [18]) and by the class of suspension (semi)flows of maps with a probability measure satisfying the corresponding local product structure of Definition 2.1.

The following results are related to Problem 4 and Problem 6 of [38] and are a continuous-time counterpart of Theorem A.

**Theorem B.** *Let  $M$  be a compact Riemannian manifold,  $X^t$  be a  $C^{1+\alpha}$ -flow on  $M$  preserving a probability measure  $\mu$ . Assume that  $\mu$  is hyperbolic and has local product structure. Then, there exists an open and dense set of maps  $\mathcal{O}$  in  $C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  such that for any  $H \in \mathcal{O}$  the cocycle  $\Phi_H^t$  has at least one positive Lyapunov exponent at  $\mu$ -almost every point. Moreover, the complement is a set with infinite codimension.*

It follows from the work of Bowen, Ruelle [18] that every  $C^{1+\alpha}$  uniformly hyperbolic flow is (semi)conjugated to a suspension semiflow over a subshift of finite type with a Hölder continuous roof function. Moreover, for every Hölder continuous potential there exists a unique equilibrium state and that measure has the local product structure. So, as a consequence of the previous theorem we obtain the following:

**Corollary 2.7.** *Given a  $C^{1+\alpha}$  ( $\alpha > 0$ ) flow  $X^t: M \rightarrow M$ ,  $\Lambda$  a hyperbolic set and  $\mu$  an equilibrium state for  $X^t|_\Lambda$  with respect to an Hölder continuous potential: there is an open and dense set of (fiber-bunched) maps  $\mathcal{O}$  in  $C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  such that, if  $H \in \mathcal{O}$ , then the cocycle  $\Phi_H^t$  has at least one positive Lyapunov exponent at  $\mu$ -almost every point. Moreover, their complement is a set with infinite codimension.*

Let us make some comments on our hypothesis. First we note that in Corollary 2.7 the hyperbolic set  $\Lambda$  is not required to be locally maximal. Nevertheless, if  $\mu$  is an  $f$ -invariant probability measure with local product structure then  $\text{supp}(\mu) \subset \Lambda$  is necessarily a locally maximal hyperbolic set for  $f$ .

With respect to Theorem B our assumptions allow us to deal with suspension flows of non-uniformly quadratic maps as Benedicks-Carleson quadratic maps, Manneville-Pommeau transformations, Hénon maps or Viana maps, and also with the geometric Lorenz and Rovella-like attractors. Thus, the majority of Hölder continuous cocycles over the flows discussed above have at least one positive Lyapunov exponent. Let us mention that Fanae [23] proved recently that an open and dense set of fiber-bunched  $sl(d, \mathbb{K})$ -cocycles over Lorenz flows have simple spectrum. Although this result should probably extend to arbitrary suspension flows with hyperbolic countable Markov structure, the fiber-bunching assumption is crucial in the argument.

In comparison one obtains the existence of at least one positive Lyapunov exponent in the symplectic case  $\mathfrak{sp}(2\ell, \mathbb{K})$  under weaker assumptions, and our theorems are stated with respect to open and dense set of infinitesimal generators while in [23] the author uses a stronger topology on the space of linear differential systems imposing a strong definition of domination. In particular, our results extend the ones of [23] for Hölder cocycles with values in  $sl(2, \mathbb{K}) = sp(2, \mathbb{K})$  over the Lorenz flow. Finally, let us mention that our proofs rely on perturbative techniques on the space of infinitesimal generators.

### 3. PRELIMINARIES

**3.1. The symplectic group of matrices.** We collect some necessary preliminary results on symplectic structures. Let  $\omega$  be a symplectic form, i.e., a closed and nondegenerate 2-form. A linear automorphism  $A: (V, \omega) \rightarrow (V, \omega)$  in a symplectic vector space  $V$  is called *symplectic* if  $A^*\omega = \omega$ , that is

$$\omega(u, v) = \omega(A(u), A(v)) \text{ for all } u, v \in V. \quad (3.1)$$

Clearly  $\dim(V) = 2\ell$  for some  $\ell \geq 1$  and the  $\ell$ -times wedging  $\omega \wedge \omega \wedge \dots \wedge \omega$  is a volume-form (see e.g. [35, Lemma 1.3]). We identify the symplectic linear automorphisms with the set of matrices and denote by  $sp(2\ell, \mathbb{R})$  ( $\ell \geq 1$ ), the non-compact  $\ell(2\ell + 1)$ -dimensional Lie group of  $2\ell \times 2\ell$  matrices  $A$  and with real entries satisfying  $A^T J A = J$ , where

$$J = \begin{pmatrix} 0 & -\mathbf{1}_\ell \\ \mathbf{1}_\ell & 0 \end{pmatrix} \quad (3.2)$$

denotes the skew-symmetric matrix,  $\mathbf{1}_\ell$  is the  $\ell$ -dimensional identity matrix and  $A^T$  stands for the transpose matrix of  $A$ .

Given a subspace  $S \subset V$ , where  $\dim(V) = 2\ell$ , we denote its  $\omega$ -orthogonal complement by  $S^\perp$  which is defined by those vectors  $u \in V$  such that  $\omega(u, v) = 0$ , for all  $v \in S$ . Clearly  $\dim(S^\perp) = 2\ell - \dim(S)$ . When, for a given subspace  $S \subset V$ , we have that  $\omega|_{S \times S}$  is non-degenerate (say  $S^\perp \cap S = \{\vec{0}\}$ ), then  $S$  is said to be a *symplectic subspace*. On the other hand, when  $\omega|_{S \times S} = 0$  (or  $S \subset S^\perp$ ) we say that the subspace  $S$  is *isotropic*. Finally, *Lagrangian* subspaces  $S$  are isotropic subspaces such that  $\dim(S) = \ell$  or, in other words,  $S^\perp = S$ , i.e. these subspaces are maximal subspaces such that the form  $\omega$  degenerates when restricted to them. We say that the basis  $\{e_1, \dots, e_\ell, e_{\hat{1}}, \dots, e_{\hat{\ell}}\}$  is a *symplectic base* of  $\mathbb{R}^{2\ell}$  if  $\omega(e_i, e_j) = 0$ , for all  $j \neq \hat{i}$  and  $\omega(e_i, e_{\hat{i}}) = 1$ .

**3.2. The symplectic geometry of Oseledets' spaces.** We now present the main geometric properties of the subspaces given by the Oseledets theorem. Through this section consider  $f \in \text{Diff}^1(M)$  and let  $\mu$  be an  $f$ -invariant probability measure. Moreover, let  $F_A$  be the cocycle over  $f$  induced by  $A \in C^{r,v}(M, sp(2\ell, \mathbb{R}))$ .

**Lemma 3.1.** *Assume that  $x$  is an Oseledets  $\mu$ -regular point with  $2\ell$  distinct Lyapunov exponents and with Oseledets decomposition in one-dimensional subspaces*

$$\mathbb{R}^{2\ell} = E_x^1 \oplus E_x^2 \oplus \dots \oplus E_x^\ell \oplus E_x^{\hat{\ell}} \oplus \dots \oplus E_x^{\hat{2}} \oplus E_x^{\hat{1}}. \quad (3.3)$$

*Then, there exists a symplectic basis  $\{e_1, \dots, e_\ell, e_{\hat{1}}, \dots, e_{\hat{\ell}}\}$  in the fiber over  $x$  formed by the invariant directions given by (3.3). Furthermore, the two-dimensional subspace  $E^i \oplus E^{\hat{i}}$  is symplectic.*

*Proof.* Let  $\{\lambda_i : i = 1 \dots 2\ell\}$  be the set of Lyapunov exponents for the cocycle  $F_A$  at the point  $x$  and, for each  $i$ , let  $E^i$  denote the corresponding one-dimensional Oseledets invariant direction. We proceed to prove that there exists a symplectic basis formed by vectors in the  $2\ell$  one-dimensional invariant subspaces in (3.3). Fix  $\lambda_i$  and  $\lambda_j$  and the correspondent invariant directions  $E^i$  and  $E^j$ , which we divide in cases. Firstly, if  $\lambda_i + \lambda_j < 0$  then for every  $\varepsilon \in (0, |\lambda_i + \lambda_j|/2)$ ,  $u_i \in E^i$  and  $u_j \in E^j$ , it follows from the theory of nonuniform hyperbolicity that there exists  $K_\varepsilon > 0$  (depending only on  $x$ ) such that for all  $n \in \mathbb{N}$

$$K_\varepsilon^{-1} e^{(\lambda_\sigma - \varepsilon)n} \|u_\sigma\| \leq \|A^n(x) \cdot u_\sigma\| \leq K_\varepsilon e^{(\lambda_\sigma + \varepsilon)n} \|u_\sigma\|$$

where  $\sigma = i, j$ . Using Cauchy-Schwartz inequality and that  $A$  is symplectic we obtain that

$$\begin{aligned} |\omega(u_i, u_j)| &= |\omega(A^n(x) \cdot u_i, A^n(x) \cdot u_j)| \leq \|A^n(x) \cdot u_i\| \|A^n(x) \cdot u_j\| \\ &\leq K_\varepsilon^2 e^{(\lambda_i + \lambda_j + 2\varepsilon)n} \|u_i\| \|u_j\|, \end{aligned}$$

which converges to zero as  $n \rightarrow +\infty$ , proving that  $\omega(u_i, u_j) = 0$ . A completely analogous reasoning for the symplectic action  $A^{-1}$  shows that if  $\lambda_i + \lambda_j > 0$ ,  $u_i \in E^i$  and  $u_j \in E^j$  and  $\varepsilon > 0$  is small there exists  $K_\varepsilon > 0$  so that

$$\begin{aligned} |\omega(u_i, u_j)| &= |\omega(A^{-n}(x) \cdot u_i, A^{-n}(x) \cdot u_j)| \leq \|A^{-n}(x) \cdot u_i\| \|A^{-n}(x) \cdot u_j\| \\ &\leq K_\varepsilon^2 e^{(-\lambda_i - \lambda_j + 2\varepsilon)n} \|u_i\| \|u_j\|, \end{aligned}$$

which also tends to zero as  $n \rightarrow +\infty$ . This also proves that  $\omega(u_i, u_j) = 0$ .

Finally, it remains the case that  $\lambda_i = -\lambda_j$ . If the two-dimensional space  $E^i \oplus E^j$  is isotropic then we change  $\lambda_j$  by any of the  $2\ell - 1$  remaining Lyapunov exponents and proceed as above. Since  $\omega$  is nondegenerated there exists some  $j'$  such that  $E^i \oplus E^{j'}$  is a symplectic two-dimensional subspace. Hence  $\omega(u_i, u_{j'}) \neq 0$  for all  $u_i \in E^i$  and  $u_{j'} \in E^{j'}$ . Hence, up to normalization and denoting  $\hat{i} = j'$ , we get  $\omega(e_i, e_{\hat{i}}) = 1$ . If one proceeds analogously and reorganize the Lyapunov exponents we obtain  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0 \geq \lambda_{\hat{\ell}} \geq \dots \geq \lambda_{\hat{2}} \geq \lambda_{\hat{1}}$  and  $\ell$  symplectic subspaces  $E^i \oplus E^{\hat{i}}$ . This completes the proof of the lemma.  $\square$

**Lemma 3.2.** *Assume that  $x$  is an Oseledets  $\mu$ -regular point with some zero Lyapunov exponent. Then, the associated invariant Oseledets subspace corresponding to the zero Lyapunov exponent has even dimension and it is symplectic.*

*Proof.* Let  $E^0$  be the subspace associated to  $\lambda = 0$ . By symmetry of the Lyapunov spectrum of  $A$  we get that  $\dim(E^0)$  is even, say equal to  $2k$ . To obtain that  $E^0$  is symplectic we prove that  $(E^0)^\perp \cap E^0 = \{\vec{0}\}$ . Given any non-zero vector  $e_1 \in E^0$ , then there exists  $e_{\hat{1}} \in \mathbb{R}^{2\ell}$  such that  $\omega(e_1, e_{\hat{1}}) \neq 0$ , because otherwise the form would be degenerate. Clearly,  $e_{\hat{1}} \in E^0$ . We complete (cf. [35, Lemma 1.2]) the symplectic base of  $E^0$  obtaining  $\{e_1, \dots, e_k, e_{\hat{1}}, \dots, e_{\hat{k}}\}$ . Let  $u \in (E^0)^\perp \cap E^0$ , then since  $u \in E^0$ ,  $u = \sum_{i=1}^k \alpha_i e_i + \beta_i e_{\hat{i}}$  for some  $\alpha_i, \beta_i \in \mathbb{R}$ . In one hand we have  $\omega(u, e_i) = \omega(\sum_{i=1}^k \alpha_i e_i + \beta_i e_{\hat{i}}, e_i) = -\beta_i$  and on the other hand we also have  $\omega(u, e_i) = \omega(\sum_{i=1}^k \alpha_i e_i + \beta_i e_{\hat{i}}, e_i) = \alpha_i$ . Moreover, since  $u \in (E^0)^\perp$  we must have  $\alpha_i = \beta_i = 0$  and so  $u = 0$ .  $\square$

In overall, given  $A \in C^{r,v}(M, sp(2\ell, \mathbb{R}))$  over a  $\mu$ -invariant diffeomorphism  $f: M \rightarrow M$ ,  $x$  an Oseledets regular point displaying  $2k \leq 2\ell$  distinct Lyapunov exponents  $\lambda_1, \dots, \lambda_k, \lambda_{\hat{k}}, \dots, \lambda_{\hat{1}}$  and with associated invariant 1-dimensional subspaces  $E^1, \dots, E^k, E^{\hat{k}}, \dots, E^{\hat{1}}$ , then

- (i) the subspaces  $E^1 \oplus \dots \oplus E^k$  and  $E^{\hat{k}} \oplus \dots \oplus E^{\hat{1}}$  are isotropic;
- (ii) the dimension of  $E^0$  is equal to  $2(\ell - k)$ , and the invariant subspace can be decomposed into  $\ell - k$  two-dimensional symplectic subspaces;
- (iii) the space  $E^0$  is symplectic;
- (iv) each two-dimensional subspaces  $E^i \oplus E^{\hat{i}}$  for  $i = 1, \dots, 2k$  are symplectic and
- (v) each subspace  $E^+ := E^1 \oplus \dots \oplus E^k$  and  $E^- := E^{\hat{k}} \oplus \dots \oplus E^{\hat{1}}$  is Lagrangian if  $k = \ell$  (see [13, Lemma 2.4 (1)]).

Let us mention that if  $A \in sp(2\ell, \mathbb{R})$  is an automorphism, then analogous conclusions of both Lemma 3.1 and Lemma 3.2 can be deduced.

**3.3. Hyperbolic and suspension flows.** In this subsection we recall some preliminaries on suspension (semi)flows and discuss the local product structure for invariant measures.

**3.3.1. Definition.** Assume that  $M_0$  is a compact metric space and that  $f : M_0 \rightarrow M_0$  is measurable. Given an  $f$ -invariant probability measure  $\mu$  and a nonzero roof (or ceiling) function  $\varrho : M_0 \rightarrow [0, +\infty)$  satisfying  $\varrho \in L^1(\mu)$  we define the *suspension semiflow*  $(X^t)_{t \geq 0}$  over  $f$  as given by  $X^t(x, s) = (x, s + t)$  acting on the space

$$M = \{(x, t) \in M_0 \times \mathbb{R}_0^+ : 0 \leq t \leq \varrho(x)\} / \sim,$$

where  $(x, \varrho(x)) \sim (f(x), 0)$ . In these coordinates  $(X^t)_t$  coincides with the flow consisting in the displacement along the “vertical” direction. If  $f$  is invertible it is not difficult to check that  $(X^t)_t$  defines indeed a flow, moreover  $(X^t)_t$  preserves the probability measure  $\hat{\mu} = (\mu \times \text{Leb}) / \int \varrho d\mu$ . Furthermore, observe that  $\hat{\mu}$  is uniquely defined by the previous expression as long as the roof function  $\varrho$  is bounded away from zero.

**3.3.2. Hyperbolic flows.** Let  $M$  be a compact and boundaryless Riemannian manifold and  $X^t : M \rightarrow M$  a smooth flow. Let also  $\Lambda \subseteq M$  be a compact and  $X^t$ -invariant set. We say that a flow  $X^t : \Lambda \rightarrow \Lambda$  is *uniformly hyperbolic* if there exists a  $DX^t$ -invariant and continuous splitting  $T_\Lambda N = E^- \oplus X \oplus E^+$  and constants  $C > 0$  and  $0 < \theta_1 < 1$  such that

$$\|DX^t|E^-\| \leq C\theta_1^t \quad \text{and} \quad \|(DX^t)^{-1}|E^+\| \leq C\theta_1^t, \quad \forall t \geq 0$$

for every  $x \in M$ . Uniformly hyperbolic flows have been well studied since the 1970’s and, in particular, their geometric structure is very well understood. It follows from the work of Bowen, Sinai and Ruelle [18, 17, 36] that hyperbolic flows admit finite Markov partitions and that are semi-conjugated to suspension flows over a hyperbolic map: there exists a subshift of finite type  $T : \Sigma \rightarrow \Sigma$ , an Hölder continuous roof function  $\varrho : \Sigma \rightarrow \mathbb{R}$  and an Hölder continuous surjective transformation  $p : \Lambda(T, \tau) \rightarrow M$  such that  $(X^t)_t$  is semi-conjugated to the suspension flow  $G^t : \Lambda(T, \tau) \rightarrow \Lambda(T, \tau)$  given by  $G^t(x, s) = (x, t + s)$ , where

$$\Lambda(T, r) = \{(x, t) : x \in \Sigma, 0 \leq t \leq \varrho(x)\} / \sim,$$

and  $\sim$  is an equivalence relation that identifies the pairs  $(x, \varrho(x))$  and  $(T(x), 0)$ . In fact,  $X^t \circ p = p \circ G^t$  for every  $t$ , the cubes  $C_i = p((x, t) : x \in [0; i], 0 \leq t \leq \varrho(x))$  are proper sets, and  $p$  is injective in a residual subset with full measure with respect to any open invariant probability measure. We say that the flow  $(X^t)_t$  exhibits a Markov partition  $\mathcal{R}$  whose rectangles are given by  $R_i = p([0; i] \times \{\vec{0}\})$ . We refer the reader to Chernov’s expository paper [20] for a detailed explanation of the theory.

In addition, it follows from the thermodynamical formalism for hyperbolic flows established in the mid 1970’s (see [18]) that there exists a unique equilibrium state  $\mu_\varphi$  with respect to any fixed Hölder continuous potential  $\varphi : \Lambda \rightarrow \mathbb{R}$ . Moreover,  $\mu_\varphi$  is obtained as a suspension of a  $T$ -invariant measure  $\mu_T$  with the local product structure (recall Definition 2.1 above).

**3.4. Regularity and Lyapunov exponents of induced cocycles.** In this subsection we describe the topology in the class of Hamiltonian linear differential systems and recall both the regularity and the Lyapunov spectrum of the time- $t$  map associated to the solution of the Hamiltonian linear system. We endow  $C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  with the  $C^{r,\nu}$ -topology defined using the norm

$$\|H\|_{r,\nu} = \sup_{0 \leq j \leq r} \sup_{x \in M} \|D^j H(x)\| + \sup_{x \neq y} \frac{\|H(x) - H(y)\|}{\|x - y\|^\nu},$$

where  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  and  $x, y \in M$ .

We introduce some necessary notions. As in the discussion in the discrete-time setting we will consider the case that  $\nu = 1$ , that is  $H$  is Lipschitz. Our starting point is to obtain that the cocycle  $\Phi_H^t(x)$  is also Lipschitz continuous.

**Lemma 3.3.** *Given any  $t \in \mathbb{R}$  and  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  there exists  $C_1 = C_1(t, H) > 0$  such that, for all  $y, z \in M$ , we have  $\|\Phi_H^t(y) - \Phi_H^t(z)\| \leq C_1 d(y, z)$ .*

*Proof.* Assume for simplicity that  $t > 0$  the computation for a negative iterate  $t$  is analogous. Since  $\Phi_H^t(x)$  is a solution of the non-autonomous ordinary differential equation  $u' = H(X^t(x)) \cdot u$  we obtain that

$$\Phi_H^t(x)v = v + \int_0^t H(X^s(x))\Phi_H^s(x)v ds,$$

for all  $t$ . Therefore, it follows from Gronwall's inequality (see e.g. [34]) that  $\|\Phi_H^t(x)v\| \leq e^{\|H\|\|t\|}\|v\|$  for all  $t \in \mathbb{R}$  and  $v \in \mathbb{R}^{2\ell}$ . Moreover, since  $H$  is Lipschitz there exists  $K > 0$  so that

$$\begin{aligned} \|\Phi_H^t(y)v - \Phi_H^t(z)v\| &\leq \int_0^t \|H(X^s(y)) - H(X^s(z))\| \|\Phi_H^s(y)v\| + \|H\| \|\Phi_H^s(y)v - \Phi_H^s(z)v\| ds \\ &\leq e^{\|t\|\|H\|}\|v\|K \int_0^t e^{-\tau s} d(y, z) ds + \int_0^t \|H\| \|\Phi_H^s(y)v - \Phi_H^s(z)v\| ds \\ &\leq e^{\|t\|\|H\|}\|v\|K\tau^{-1}d(y, z) + \int_0^t \|H\| \|\Phi_H^s(y)v - \Phi_H^s(z)v\| ds \end{aligned}$$

Applying again Gronwall's lemma it follows that  $\|\Phi_H^t(y)v - \Phi_H^t(z)v\| \leq e^{2\|t\|\|H\|}\|v\|K d(y, z)$  and, consequently, we obtain that for all  $y, z \in M$

$$\|\Phi_H^t(y) - \Phi_H^t(z)\| \leq e^{2\|t\|\|H\|}K d(y, z),$$

which proves the lemma.  $\square$

The next result relates the Lyapunov spectrum of cocycles with the induced one.

**Lemma 3.4.** *Given  $f : M \rightarrow M$ , an  $f$ -invariant, ergodic probability measure  $\mu$  and a roof function  $\varrho : M \rightarrow [0, +\infty)$  with  $\varrho \in L^1(\mu)$  let  $X^t$  be the suspension flow and  $\hat{\mu} = (\mu \times \text{Leb}) / \int \varrho d\mu$  be an invariant probability measure. Given  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  the cocycle  $(\Phi_H^t, \hat{\mu})$  has a non-zero Lyapunov exponent if and only if the same property holds for the cocycle  $(\Psi_H, \mu)$  over  $f$  given by  $\Psi_H(x) = \Phi_H^{\varrho(x)}(x)$ .*

*Proof.* Since  $\mu$  is ergodic, then the  $X^t$ -invariant probability measure  $\hat{\mu}$  is also ergodic and the largest Lyapunov exponent  $\lambda^+(\Phi_H^t, \hat{\mu})$  for the time continuous cocycle  $\Phi_H^t$  is given by

$$\lambda^+(\Phi_H^t, \hat{\mu}) = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\Phi_H^t(x)\| \quad \text{for } \hat{\mu}\text{-a.e. } x.$$

Moreover, by construction of  $\hat{\mu}$ , it follows from the Birkhoff ergodic theorem and ergodicity of  $\mu$  that

$$\lim_{n \rightarrow \pm\infty} \frac{\varrho^{(n)}(z_0)}{n} = \int \varrho \, d\mu \quad \text{and} \quad \lambda^+(\Psi_H, \mu) = \lim_{k \rightarrow \pm\infty} \frac{1}{k} \log \|\Psi_H^k(z_0)\|$$

for  $\mu$ -almost every  $z_0 \in M \times \{0\}$ , where

$$\varrho^{(n)}(x) = \sum_{0 \leq j \leq n-1} \varrho(f^j(x)). \quad (3.4)$$

In particular, since  $\varrho(z)$  denotes the first Poincaré hitting time of a point  $z \in \hat{M}$  to the global section  $M \times \{0\}$  then for  $\mu$ -almost every  $z \in \hat{M}$  one has that  $z_0 = X^{\varrho(z)}(z)$  satisfies

$$\begin{aligned} \lambda^+(\Phi_H^t, \mu) &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\Phi_H^{t-\varrho(z)}(X^{\varrho(z)}(z)) \Phi_H^{\varrho(z)}(z)\| \\ &= \lim_{n \rightarrow \pm\infty} \frac{1}{\varrho^{(n)}(z_0)} \log \|\Phi_H^{\varrho^{(n)}(z_0)}(z_0)\|. \\ &= \left( \lim_{n \rightarrow \pm\infty} \frac{n}{\varrho^{(n)}(z_0)} \right) \left( \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\Psi_H^n(z_0)\| \right) \\ &= \left( \int \varrho \, d\mu \right)^{-1} \lambda^+(\Psi_H, \mu), \end{aligned}$$

where  $\lambda^+(\Psi_H, \mu)$  denotes the maximum Lyapunov exponent of the cocycle  $\Psi_H$  with respect to  $\mu$ . In particular the largest Lyapunov exponent of  $\Phi_H^t$  with respect to  $\mu$  is zero if and only if the same holds for  $\Psi_H$  with respect to  $\mu$ . Similar computations prove the same for the lowest Lyapunov exponent

$$\lambda^-(\Phi_H^t, \mu) = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\Phi_H^t(x)^{-1}\|^{-1} \quad \text{for } \hat{\mu}\text{-a.e. } x.$$

This finishes the proof of the lemma.  $\square$

#### 4. DISCRETE-TIME SYMPLECTIC COCYCLES

**4.1. A quick tour on the proof of Theorem A.** In this section we prove Theorem A for symplectic cocycles over non-uniformly hyperbolic maps. Let  $\mu$  be an  $f$ -invariant, ergodic and hyperbolic measure. We proceed as described at the end of Section 2.

First we describe the strategy, that goes along some ideas developed in [16, 38] for discrete time cocycles over non-uniformly hyperbolic maps in the more subtle symplectic perturbation theory. Consider a  $C^{1+\alpha}$ -diffeomorphism  $f$  preserving an  $f$ -invariant, ergodic and hyperbolic measure with local product structure.



By non-uniform hyperbolicity of  $(f, \mu)$  there are, not necessarily invariant, compact Pesin sets  $\mathcal{H}(K, \tau)$  of points with uniform hyperbolicity constants along the stable and unstable Pesin submanifolds. Then, if the cocycle  $F_A$  associated to  $A \in C^{r, \nu}(M, sp(2\ell, \mathbb{K}))$  has only zero Lyapunov exponents then there exist compact sets (i.e. domination blocks cf. §4.2) where the cocycle behaves like a partially hyperbolic map in a sense that the fiber behaves like a central manifold which is dominated by the base dynamics. In consequence, there are well defined stable holonomy transformations

$$\begin{aligned} H_{x,y}^s &:= H_{A,x,y}^s: \{x\} \times \mathbb{K}^{2\ell} \longrightarrow \{y\} \times \mathbb{K}^{2\ell} \\ v &\longrightarrow \left[ \lim_{n \rightarrow +\infty} A^{-n}(f^n(y))A^n(x) \right] \cdot v, \end{aligned}$$

for points  $y \in W^s(x)$ , which correspond to central manifolds to the partially hyperbolic dynamics in  $F_A$  and are linear symplectic transformations. Unstable holonomies  $H_{x,y}^u$  are defined analogously using unstable manifolds and  $f^{-1}$ . Let  $h_{x,y}^s$  and  $h_{x,y}^u$  denote the projectivization on the fibers of the stable and unstable holonomies respectively (see §4.2 for full details). Furthermore, given any  $f_A$ -invariant probability measure  $m$  such that  $\Pi_* m = \mu$  (recall  $\mu$  has local product structure) there exists a continuous disintegration  $(m_x)_{x \in M}$  of  $m$  such that  $(h_{x,y}^s)_* m_x = m_y$  for all  $y \in W^s(x)$  and  $(h_{x,y}^u)_* m_x = m_y$  for all  $y \in W^u(x)$  with  $x$  belonging to the holonomy block. Moreover, such property holds for all periodic points  $p_1, p_2, \dots, p_k$  in the domination block that are homoclinically related, meaning that there exists

$$z_i \in W_{\text{loc}}^u(p_i) \cap W_{\text{loc}}^s(p_{i+1}) \quad \text{for } i = 1 \dots k-1.$$

The existence of such periodic points in domination blocks was guaranteed in [38]. Thus, if  $A$  has only zero Lyapunov exponents and  $\{p_i\}_{i=1}^k$  are periodic points whose Lyapunov exponents are all distinct then

$$(h_{A,p_i,z_i}^u)_* m_{p_i} = m_{z_i} = (h_{A,p_{i+1},z_i}^s)_* m_{p_{i+1}} \quad \text{for all } i = 1 \dots k-1, \quad (4.1)$$

and is a finite convex combination of Dirac measures. Hence, we are reduced to show that the condition described in (4.1) is a highly non-generic condition on  $A$ . This was done for  $sl(d, \mathbb{K})$ -cocycles by Viana using an elegant argument to prove that the map  $A \mapsto H_{A,x,y}$  is a submersion and, consequently, the set of cocycles  $A \in C^{r, \nu}(M, sl(d, \mathbb{K}))$  satisfying (4.1) is contained in a closed subset of empty interior and has infinite codimension. As pointed out by Viana [38, page 678] this argument fails to extend to groups of matrices with dimension smaller than  $2\ell(2\ell-1)$ , as the symplectic group  $sp(2\ell, \mathbb{K})$ . To overcome these difficulties, in §4.4, we use a symplectic perturbative approach in very small neighborhoods of heteroclinic points to show that every cocycle  $A$  is  $C^{r+\nu}$ -approximated by open sets of cocycles so that unstable holonomies remain unchanged while stable holonomies are modified in order not to satisfy the rigid condition in (4.1). This finishes the sketch of the proof.

**4.2. Zero Lyapunov exponents lead to rigidity.** In this subsection we shall collect some ingredients from [38] and show that cocycles  $A$  whose Lyapunov exponents *are all zero* exhibit a rigid condition for all  $f_A$ -invariant measures on the fibered projective space. Recall the following definition.

**Definition 4.1.** Let  $A \in C^0(M, sp(2\ell, \mathbb{K}))$  be a continuous cocycle. Given  $N \geq 1$  and  $\theta > 0$ , consider the set  $\mathcal{D}_A(N, \theta)$  of points  $x \in M$  satisfying

$$\prod_{j=0}^{k-1} \|A^N(f^{jN}(x))\| \|A^N(f^{jN}(x))^{-1}\| \leq e^{kN\theta} \quad \text{for all } k \in \mathbb{N}.$$

For simplicity we say that  $\mathcal{O}$  is a *holonomy block* for  $A$  if it is a compact subset of  $\mathcal{H}(K, \tau) \cap \mathcal{D}_A(N, \theta)$  for some constants  $K, \tau, N, \theta$  satisfying  $3\theta < \tau$ . This property means that the cocycle  $f_A$  behaves like a partially hyperbolic dynamics with the central direction corresponding to the fibers, leading to strong-stable and strong-unstable foliations. Moreover, since domination is an open condition for the cocycle this enables to obtain strong-stable and strong-unstable foliations for all nearby cocycles. More, precisely,

**Proposition 4.2.** *For every  $x \in \mathcal{O}$  and  $y \in W_{loc}^u(x)$ , there exists  $C_1 > 0$  and a symplectic linear transformation  $H_{A,x,y}^u : \{x\} \times P\mathbb{K}^{2\ell} \rightarrow \{y\} \times P\mathbb{K}^{2\ell}$  such that:*

- (1)  $H_{x,x}^u = id$  and  $H_{x,z}^u = H_{y,z}^u \circ H_{x,y}^u$ ;
- (2)  $A(f^{-1}(y)) \circ H_{f^{-1}(x), f^{-1}(y)}^u \circ A(x)^{-1} = H_{x,y}^u$ ;
- (3)  $\|H_{x,y}^u - id\| \leq C_1 d(x, y)$  and
- (4)  $H_{f^j(y), f^j(z)}^u = A^j(z) \circ H_{y,z}^u \circ A^j(y)^{-1}$  for all  $j \in \mathbb{Z}$ ,

for every  $x, y, z$  in the same local unstable manifold.

*Proof.* This is a consequence of [16, Proposition 1.2]. In fact, since the cocycle  $F_A$  varies Lipschitz continuously on the fibers and satisfies the fiber-bunched property in the domination blocks, then it follows by [15, 16, 38] that the limit

$$H_{x,y}^u = \lim_{n \rightarrow \infty} [A^{-n}(y)]^{-1} A^{-n}(x) = \lim_{n \rightarrow \infty} A^n(f^{-n}(y)) A^{-n}(x) \quad (4.2)$$

does exist for every  $y \in W^u(x)$ . Since  $\{A^n(f^{-n}(y)) A^{-n}(x)\}_{n \in \mathbb{N}}$  is a Cauchy sequence in the complete metric space  $sp(2\ell, \mathbb{K})$  then it follows that  $H_{x,y}^u$  defines a symplectic linear map. Properties (1) and (2) are immediate from the definition while property (3) is a consequence of Proposition 1.2 in [16]. Finally, property (4) follows easily from (4.2). This finishes the proof of the proposition.  $\square$

Notice that each cocycle  $A$  induces a projectivized cocycle  $f_A : M \times P\mathbb{K}^{2\ell} \rightarrow M \times P\mathbb{K}^{2\ell}$  with compact fiber  $P\mathbb{K}^{2\ell}$ . In particular, the set of  $f_A$ -invariant probability measures is non-empty by the Krylov-Bogoliubov theorem. Let  $h_{A,x,y}^u$  (respectively  $\{h_{A,x,y}^s\}$ ) be the transformations obtained from  $H_{A,x,y}^u$  (respectively  $\{H_{A,x,y}^s\}$ ) by projectivization on the fibers and let us refer as *unstable (respectively stable) holonomies* for the projectivized cocycle  $f_A$ . We omit its dependence on  $A$  for notational simplicity when no confusion is possible. Given a holonomy block  $\mathcal{O}$  and  $\delta > 0$  small enough (depending only on  $K, \tau$ ), for all  $x \in \text{supp}(\mu \mid \mathcal{O})$  let  $\mathcal{N}_x(\mathcal{O}, \delta)$ ,  $\mathcal{N}_x^u(\mathcal{O}, \delta)$  and  $\mathcal{N}_x^s(\mathcal{O}, \delta)$  be the induced neighborhoods of  $x$  in  $\mathcal{O}$  with local product structure defined similarly as in §2.1.

We say that an  $f_A$ -invariant probability measure  $m$  admits a *continuous disintegration* on  $\tilde{M} \subset M$  if  $m(A) = \int m_x(A) d\mu(x)$  for all measurable subset  $A \subset M \times P\mathbb{K}^{2\ell}$  and  $\tilde{M} \ni x \mapsto m_x$  is continuous in the weak\* topology. The following result assert roughly that any  $f_A$ -invariant

probability measure  $m$  such that  $\Pi_* m = \mu$  admits such disintegration for all points in the support of holonomy blocks.

**Proposition 4.3.** [38, Proposition 3.5] *Let  $O$  be a positive  $\mu$ -measure holonomy block, consider  $x \in \text{supp}(\mu \mid O)$  and set the neighborhoods  $N_x(O, \delta)$  of  $x$  as above. Then, every  $f_A$ -invariant probability measure  $m$  with  $\Pi_* m = \mu$  admits a continuous disintegration on  $\text{supp}(\mu \mid N_x(O, \delta))$ . Moreover,*

$$m_z = (h_{y,z}^s)_* m_y \quad \text{and} \quad m_z = (h_{w,z}^u)_* m_w$$

*for all  $y, z, w \in \text{supp}(\mu \mid N_x(O, \delta))$  such that  $y, z$  belong to the same strong-stable local manifold and  $z, w$  belong to the same strong-unstable local manifold.*

Moreover, an analogous result yields invariance of the disintegrated measures by unstable holonomies. Therefore, the purpose is to consider periodic dominated points that are homoclinically related, where the characterization of the disintegrated measures  $m_z$  is simple. This is done in the following subsection.

**4.3. Obstructions using periodic points.** In this section we collect some results on the obstruction to zero Lyapunov exponents obtained using heteroclinic orbits associated to periodic points. Throughout this section let  $O$  be a positive  $\mu$ -measure holonomy block for a cocycle  $A$  such that  $\lambda^+(A, \mu) = 0$  and let  $x \in \text{supp}(\mu \mid O)$  be as above. We begin with the following result.

**Lemma 4.4.** *If  $m$  is an  $f_A$ -invariant probability measure such that  $\Pi_* m = \mu$  and  $p \in \text{supp}(\mu \mid N_x(O, \delta))$  is a periodic point of period  $\pi$  for  $f$ , then  $A_*^\pi m_p = m_p$ . Moreover, if  $A^\pi$  has all real and distinct eigenvalues, then there exist elements  $\{v_i\}_{i=1 \dots 2\ell}$  in  $P\mathbb{K}^d$  and a probability vector  $\{\alpha_i\}_{i=1 \dots 2\ell}$  such that  $m_p = \sum_{i=1}^{\ell} \alpha_i \delta_{v_i}$ .*

*Proof.* Let  $m$  be an arbitrary  $f_A$ -invariant probability measure and  $(m_x)_x$  be a continuous disintegration of  $m$  on  $\text{supp}(\mu \mid N_x(O, \delta))$ . Since  $m = \int m_x d\mu(x)$  and  $(f_A)_* m = m$  then, for all  $k$ , we have  $A^k(x)_* m_x = m_{f^k(x)}$  for  $\mu$ -almost every  $x$ . Hence, by continuity in the weak\* topology at  $p \in \text{supp}(\mu \mid N_x(O, \delta))$  it follows that  $A_*^\pi m_p = m_p$ , which proves the first claim. Now, if  $\{v_i\}_{i=1 \dots 2\ell}$  are linearly independent unitary eigenvectors of  $A^\pi$  in  $\mathbb{K}^{2\ell}$  then every such vectors are the unique fixed points for the Morse-Smale action of  $A^\pi$  on the projective space. Thus, every  $A^\pi$ -invariant probability is a convex combination of Dirac measures at the points  $\{v_i\}_{i=1 \dots 2\ell}$ . This finishes the proof of the lemma.  $\square$

Following [38], given a periodic point with hyperbolicity constants  $K, \tau$  we say that  $p$  is *dominated* if there exists  $P \geq 1$  such that  $p \in \mathcal{D}_A(P\pi, \theta)$ , where  $\pi$  is the period of  $p$  and  $3\theta < \tau$ . In particular it follows that

$$H_{A,p,z}^s = \lim_{n \rightarrow \infty} A^{\pi n}(z)^{-1} A^{\pi n}(p)$$

defines the stable holonomies for all points in the local stable manifold of  $p$ , and similarly using  $f^{-1}$  for unstable holonomies. We say that a periodic point is *simple* if it has all eigenvalues of different norm. The following is a direct consequence of Proposition 4.3.

**Corollary 4.5.** *Let  $m$  be an  $f_A$ -invariant probability measure such that  $\Pi_* m = \mu$  and assume that  $p, q \in \text{supp}(\mu \mid N_x(O, \delta))$  are dominated periodic points for  $f$  and  $z$  is the unique point in the heteroclinic intersection  $W_{loc}^u(q) \cap W_{loc}^s(p)$ . Then*

$$m_z = (h_{A,q,z}^u)_* m_q = (h_{A,p,z}^s)_* m_p.$$

The former corollary will be of particular interest in the case that  $p$  and  $q$  are simple periodic points. In fact, if this is the case, while on the one hand  $m_z$  is an atomic measure whose atoms are Dirac measures at the vectors  $h_{A,p,z}^s(v_i)$  where  $v_i$  are the eigenvectors for  $A^{\pi_1}(p)$  (being  $\pi_1$  the period of  $p$ ), on the other hand it also belongs to the convex hull of the Dirac measures at points  $h_{A,q,z}^u(w_i)$  where  $w_i$  are the eigenvectors for  $A^{\pi_2}(q)$  (being  $\pi_2$  the period of  $q$ ). In the next subsection we prove that the set of cocycles satisfying these invariance-like properties is small from the topological viewpoint.

**4.4. Perturbation results.** In this section we use perturbative arguments for symplectic cocycles to prove that the holonomy invariance is a rigid condition. As we already said the technique of [38] does not hold due to lack of dimension for  $sp(2\ell, \mathbb{R}) \subset sl(2\ell, \mathbb{R})$ . First, we consider a basic perturbative result which allows us to obtain simple and real spectrum by making small perturbations once we assume that we have a periodic point with large period. Then, we show that typically simple periodic points on holonomy blocks do exist. Assume that a cocycle  $A \in C^{r,\nu}(M, sp(2\ell, \mathbb{R}))$  satisfies  $\lambda^+(A, \mu) = 0$ . For simplicity of the presentation, we consider the case that  $\ell = 2$  which in the case of the symplectic group encloses all difficulties. The general case of dimension  $2\ell$  (for  $\ell > 2$ ) is obtained by spectral decomposition and reduction to the cases discussed below.

We say that  $A \in sp(4, \mathbb{R})$  is (see Figure 1):

- (A) a *complex saddle* if it has a non real eigenvalue with norm greater than 1;
- (B) a *saddle center* if it has a non real eigenvalue with norm 1 and a real eigenvalue with norm greater than 1;
- (C) a *generic center* if it has four different non real eigenvalues with norm 1;
- (D) a *degenerated center* if it has two non real eigenvalues with norm 1 and with multiplicity two.

We consider the case when the eigenspace equals the root space in order to simplify the presentation. So, from now on we will consider that the previous types of matrices have a normal form, in the canonical symplectic basis where the form is  $\omega = e_1 \wedge \hat{e}_1 + e_2 \wedge \hat{e}_2$ , in the following form:

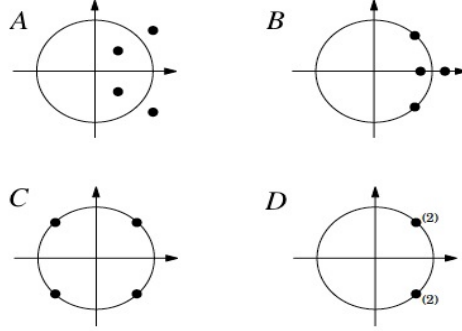


FIGURE 1. *A* complex saddle, *B* saddle center, *C* generic center and *D* degenerated center.

type	$sp(4, \mathbb{R})$ matrix	constraints	eigenvalues
complex saddle	$\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \frac{a}{a^2+b^2} & -\frac{b}{a^2+b^2} \\ 0 & 0 & \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix}$	$b \neq 0$	$a \pm bi$ $\frac{a \pm bi}{a^2+b^2}$
saddle center	$\begin{pmatrix} a & 0 & -b & 0 \\ 0 & c & 0 & 0 \\ b & 0 & a & 0 \\ 0 & 0 & 0 & c^{-1} \end{pmatrix}$	$a^2 + b^2 = 1$ $b, c \neq 0$	$a \pm bi$ $c, c^{-1}$
generic center	$\begin{pmatrix} a & 0 & -b & 0 \\ 0 & c & 0 & -d \\ b & 0 & a & 0 \\ 0 & d & 0 & c \end{pmatrix}$	$a^2 + b^2 = 1$ $c^2 + d^2 = 1$ $b, d \neq 0$	$a \pm bi$ $c \pm di$
degenerated center	$\begin{pmatrix} a & 0 & -b & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & b & 0 & a \end{pmatrix}$	$a^2 + b^2 = 1$ $b \neq 0$	$a \pm bi$

The following result will be central in order to obtain Lemma 4.7.

**Lemma 4.6.** ([14, Lemme 6.6]) *For all  $\varepsilon > 0$ , there exists  $N \geq 1$  such that, for all  $n \geq N$  and all finite collection  $\{A_0, A_1, \dots, A_{n-1}\}$  of elements in  $sp(2, \mathbb{R})$  there exists  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$  in  $] -\varepsilon, \varepsilon[$  on which the matrix  $B_{n-1} \circ \dots \circ B_1 \circ B_0$  has all its eigenvalues real, where  $B_i := R_{\alpha_i} \circ A_i$  and  $R_{\alpha_i}$  stands for the planar rotation of angle  $\alpha_i$ .*

We will now prove a symplectic perturbation argument to obtain simplicity of the Lyapunov spectrum for products of  $sp(4, \mathbb{R})$  matrices, that will be of particular interest to deal with periodic points.

**Lemma 4.7.** *For all  $\varepsilon > 0$ , there exists  $N \geq 1$  such that, for all  $n \geq N$  and all finite collection  $\{A_0, A_1, \dots, A_{n-1}\}$  of  $K$ -bounded elements in  $sp(4, \mathbb{R})$  such that if  $A_{n-1} \circ A_{n-2} \circ \dots \circ A_1 \circ A_0$  is*

- (i) *complex saddle or*
- (ii) *a saddle center or*
- (iii) *a generic center or*
- (iv) *a degenerated center, then*

there exist  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$  in  $]-\varepsilon, \varepsilon[$  satisfying the following property: For all  $i \in \{0, 1, \dots, n-1\}$ , there exists  $B_i \in sp(4, \mathbb{R})$   $3\varepsilon K$ -close to  $A_i$  such that  $B_{n-1} \circ \dots \circ B_1 \circ B_0$  has simple and real spectrum.

*Proof.* Let us prove (i). Fix  $\varepsilon > 0$ . By our assumption we have that the 2-dimensional subspace  $\mathcal{S}^1$  generated by  $e_1 = (1, 0, 0, 0)$  and  $e_2 = (0, 1, 0, 0)$  is invariant by  $A_{n-1} \circ A_{n-2} \circ \dots \circ A_1 \circ A_0$ . Moreover, the 2-dimensional subspace  $\mathcal{S}^2$  generated by  $e_3 = (0, 0, 1, 0)$  and  $e_4 = (0, 0, 0, 1)$  is also invariant by  $A_{n-1} \circ A_{n-2} \circ \dots \circ A_1 \circ A_0$ . We consider the following two families of 2-dimensional subspaces  $\{\mathcal{S}_i^j\}_{i=0}^n$ , for  $j = 1, 2$ , defined recurrently by:  $\mathcal{S}_0^j = \mathcal{S}^j$ ,  $\mathcal{S}_1^j = A_{n-1}^{-1}(\mathcal{S}_0^j)$ ,  $\mathcal{S}_2^j = A_{n-2}^{-1}(\mathcal{S}_1^j)$ , ...,  $\mathcal{S}_i^j = A_{n-i}^{-1}(\mathcal{S}_{i-1}^j)$ , ...,  $\mathcal{S}_{n-1}^j = A_1^{-1}(\mathcal{S}_{n-2}^j)$  and  $\mathcal{S}_n^j = A_0^{-1}(\mathcal{S}_{n-1}^j)$ . By invariance we have  $\mathcal{S}^j = \mathcal{S}_n^j$ . Now, we define  $R_{\alpha_i}^j \in sp(4, \mathbb{R})$  in the following way:  $R_{\alpha_i}^j$  rotates by an angle  $\alpha_i$  in the (symplectic) subspace  $\mathcal{S}_i^j$  and is the identity in its symplectic orthogonal complement. Finally, define  $B_i := R_{\alpha_i}^1 \circ R_{\alpha_i}^2 \circ A_i$ . Since  $B_i$  is the composition of three symplectic linear maps  $B_i \in sp(4, \mathbb{R})$ . Moreover, using the invariance of the two-dimensional subspaces, by a direct application of Lemma 4.6 gives that there exist  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$  in  $]-\varepsilon, \varepsilon[$  such that  $B_{n-1} \circ \dots \circ B_1 \circ B_0$  has all its eigenvalues real,  $\|R_{\alpha_i}^1 \circ R_{\alpha_i}^2 - id\| \leq 2\varepsilon$  and our perturbation transform the matrix

$$\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \frac{a}{a^2+b^2} & -\frac{b}{a^2+b^2} \\ 0 & 0 & \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} \quad \text{into the matrix} \quad \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta^{-1} & 0 \\ 0 & 0 & 0 & \eta^{-1} \end{pmatrix}.$$

In addition, for all  $i$  we have that

$$\begin{aligned} \|B_i - A_i\| &= \|R_{\alpha_i}^1 \circ R_{\alpha_i}^2 \circ A_i - A_i\| = \|(R_{\alpha_i}^1 \circ R_{\alpha_i}^2 - id) \circ A_i\| \\ &\leq \|A_i\| \|R_{\alpha_i}^1 \circ R_{\alpha_i}^2 - id\| \leq 2\varepsilon K. \end{aligned}$$

At last, up to consider the symplectic perturbation of  $B_{n-1}$  by  $S \circ B_{n-1}$  given by  $S(e_1) = e_1$ ,  $S(e_2) = (1 + \varepsilon K)e_2$ ,  $S(e_3) = e_3$  and  $S(e_4) = e_4/(1 + \varepsilon K)$  it is clear that  $\|B_i - A_i\| \leq 3\varepsilon K$  and that  $B_{n-1} \circ \dots \circ B_1 \circ B_0$  has simple real spectrum. Since the proof of items (ii), (iii) and (iv) follow a completely similar argument we leave the details for the reader.  $\square$

As pointed out before the previous lemma can be easily extended Lemma 4.7 to dimension  $2\ell$  ( $\ell > 2$ ). We make use of this remark to prove the following proposition.

**Proposition 4.8.** *Given  $\varepsilon > 0$  and  $k \geq 2$  there exists a holonomy block  $\mathcal{O}$  for  $A$  so that  $\mu(\mathcal{O}) > 1 - \varepsilon$ , distinct dominated periodic points  $\{p_i\}_{i=1}^k$  in  $\mathcal{O}$  and a cocycle  $B \in C^{r,v}(M, sp(2\ell, \mathbb{K}))$  such that the following properties hold:*

- (1)  $W_{loc}^u(p_i) \cap W_{loc}^s(p_{i+1}) \neq \emptyset$  consists of one point;
- (2)  $p_i \in \text{supp}(\mu \mid \mathcal{O} \cap f^{-\pi_i}(\mathcal{O}))$ , where  $\pi_i$  is the period of the periodic point  $p_i$ , for all  $1 \leq i \leq k$ ;

(3)  $\|A - B\|_{r,v} < \varepsilon$  and

(4) the Lyapunov spectrum of  $B^{\pi_i}(p_i)$  is real and simple.

Finally, the set of cocycles  $B$  satisfying (1), (2) and (4) is open in the  $C^{r,v}$ -topology.

*Proof.* The arguments of this proof are borrowed from results in [38, §4] together with a symplectic perturbation version of [16, Proposition 9.1] for the spectra of the periodic orbits. For that reason we just sketch the main argument for reader's convenience focusing on the symplectic perturbative argument. The sketch is somewhat more detailed in the proof of Proposition 6.7 further ahead in the flows context. Let  $\tilde{O}$  be a holonomy block for  $A$  with large measure and  $x \in \text{supp}(\mu \mid \tilde{O})$ . It follows from [38, Corollary 4.8] that there exists  $\rho > 0$  and there are  $k \geq 2$  distinct periodic points  $p_1, \dots, p_k \in B(x, \rho/2)$  of periods  $\pi_1, \dots, \pi_k$  such that:

- (a)  $\text{dist}(f^n(y), f^n(z)) \leq K e^{-\tau n} \text{dist}(y, z)$  for all  $n \geq 0$  and  $y, z \in W_{\text{loc}}^s(p_i)$ ,
- (b)  $W_{\text{loc}}^s(p_i)$  has size at least  $\rho$  and intersects every  $W_{\text{loc}}^u(p_j)$  for all  $i, j$  (and analogously for local unstable manifolds).

Moreover, following *ipsis literis* [38, §4.3] there exists a holonomy block  $O \supset \tilde{O}$  such that all periodic points belong to  $O$  and  $p_i \in \text{supp}(\mu \mid O \cap f^{-\pi_i}(O))$ . This yields properties (1) and (2) above. Thus, we are reduced to prove that there exists a symplectic cocycle  $B \in C^{r,v}(M, sp(2\ell, \mathbb{K}))$  such that  $\|A - B\|_{r,v} < \varepsilon$  and the Lyapunov spectrum of  $B^{\pi_i}(p_i)$  is real and simple for all  $i$ .

If the later property holds for the cocycle  $A$  with respect to the periodic points  $p_i$  we are done. Otherwise, up to a small perturbation we may assume that there exists a complex eigenvalue  $\sigma$  of largest norm for  $A^{\pi_i}(p_i)$ , and that the only complex eigenvalue of equal norm  $|\sigma|$  is  $\bar{\sigma}$ . Recall that it is guaranteed by the symplectic structure that  $\bar{\sigma}^{-1}$  and  $\sigma^{-1}$  are also eigenvalues (see [35, Proposition 1.5]). We will assume that they are the only complex eigenvalues, since otherwise recursive perturbations can be made to reduce the number of complex eigenvalues, and follow the strategy of [16]. It follows from the  $\lambda$ -lemma that there exist homoclinic points  $z_i$  for  $p_i$ . Consider  $\Lambda_i$  to be the hyperbolic set obtained as the maximal invariant set in a small neighborhood of  $p_i$  and  $z_i$ . Then, using the symbolic dynamics one can find a sequence of periodic points  $x_n \in \Lambda_i$  of period  $\pi(x_n)$  multiple of  $\pi_i$  that spend most of the orbit in a small neighborhood of  $p_i$  so that the complex eigenvalues of  $A^{\pi(x_n)}(x_n)$  with largest (respectively smallest) norm are close to the real number  $|\sigma|$  (respectively  $|\sigma|^{-1}$ ).

Thus using the perturbative Lemma 4.7 one can find a cocycle  $B$  that is  $C^{r,v}$ -close to  $A$  and such that  $B^{\pi(x_{n_i})}(x_{n_i})$  has simple and real Lyapunov spectrum, i.e.,  $2\ell$  real and distinct eigenvalues of different norm. On the other hand, if the neighborhood of  $p_i$  and  $z_i$  is small enough, then the local stable and unstable manifolds of the periodic points  $p_i = x_{n_i}$  obtained recursively as above are uniformly long in such a way that properties (1) and (2) still hold. Since the last assertion in the proposition is immediate this finishes its proof.  $\square$

The next crucial lemma asserts that one can perturb, in the  $C^{r,v}$ -topology, any cocycle with all its Lyapunov exponents equal to zero to show that open and densely there is no holonomy invariance of the disintegrated measures. More precisely,

**Lemma 4.9.** (*Breaking lemma*) Let  $W = \{w_i : i = 1 \dots 2\ell\}$  be any linearly independent set of vectors in the fiber  $\mathbb{K}P^{2\ell}$  over  $z \in W_{\text{loc}}^s(p) \cap W_{\text{loc}}^u(q)$ . Given  $\varepsilon > 0$ ,  $A \in C^{r,v}(M, sp(2\ell, \mathbb{K}))$

and a symplectic base  $\{v_i : i = 1 \dots 2\ell\}$  in the fiber  $\mathbb{K}P^{2\ell}$  over  $p$ , there exists a cocycle  $B \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  such that  $\|A - B\|_{r,\nu} < \varepsilon$ , the unstable holonomies coincide  $H_{B,q,z}^u = H_{A,q,z}^u$  and  $H_{B,p,z}^s(v_i)$  does not belong to the 1-dimensional subspace generated by  $w_j$  for all  $j$ . Moreover, the later property is open in the  $C^{r,\nu}$ -topology.

*Proof.* Let  $\pi_1 = \pi(p)$  and  $\pi_2 = \pi(q)$  be the periods of the periodic point  $p$  and  $q$ , respectively. Since the limits do exist recall that  $H_{A,q,z}^u = \lim_{n \rightarrow \infty} [A^{-\pi_2 n}(z)]^{-1} A^{-\pi_2 n}(q)$  and also, for all  $j \geq 0$ ,

$$H_{A,p,z}^s = \lim_{n \rightarrow \infty} A^{\pi_1 n}(z)^{-1} A^{\pi_1 n}(p) = [A^{j\pi_1}(z)]^{-1} H_{A,p,f^{j\pi_1}(z)}^s. \quad (4.3)$$

Our strategy is to perform a small symplectic perturbation on a small neighborhood  $V$  around the point  $f^{\pi_1}(z)$  such that the unstable holonomy  $H_{B,p,z}^u$  for the perturbed cocycle  $B$  remains equal while

$$H_{B,p,z}^s(v_i) = [B^{\pi_1}(z)]^{-1} H_{B,p,f^{\pi_1}(z)}^s(v_i)$$

is linearly independent with each  $w_j$  for all  $j$ , as we now detail.

Since the forward orbit of  $z$  is convergent to the one of  $p$  and the backward orbit of  $z$  is convergent to the periodic orbit of  $q$ , then there exists a neighborhood  $V$  of  $f^{\pi_1}(z)$  such that  $V \cap \{f^i(z) : i \in \mathbb{Z}\}$  consists of  $f^{\pi_1}(z)$ . We are going to construct the desired local perturbation  $B \in C^{r,\nu}(M, sp(2\ell, \mathbb{R}))$  of the cocycle  $A$  in the neighborhood  $V$ . Since all constructions are done in local charts using the  $C^\infty$  Riemannian structure of the manifold  $M$  one may assume without loss of generality that  $V \subset \mathbb{R}^d$ , where  $d = \dim(M)$ . Now, fix a small  $\delta > 0$  with  $B(f^{\pi_1}(z), \delta) \subset V$  being the support of the perturbation and such that  $B$  is  $C^{r,\nu}$ -close to  $A$  as follows.

Let  $\mathbb{K}_x^{2\ell} = E_x^1 \oplus E_x^2 \oplus \dots \oplus E_x^\ell \oplus E_x^{\hat{\ell}} \oplus \dots \oplus E_x^{\hat{2}} \oplus E_x^{\hat{1}}$  be the symplectic decomposition of  $\mathbb{K}_x^{2\ell}$  over  $x = f^{\pi_1}(z)$  cf. (3.3) into two-dimensional symplectic subspaces  $E_x^s \oplus E_x^{\hat{s}}$  induced by the vectors  $e_i = H_{A,p,f^{2\pi_1}(z)}^s(v_i)$  for  $i = 1 \dots 2\ell$ . Fix  $\varepsilon > 0$  and consider a  $C^\infty$  bump function  $\varphi : [0, +\infty[ \rightarrow [0, +\infty[$  such that  $\varphi(t) = 0$  if  $t > \delta$  and  $\varphi(t) = 1$  if  $t < \frac{\delta}{2}$ . Define  $S \in C^{r,\nu}(M, sp(2\ell, \mathbb{R}))$  so that the map  $S(y)$  rotates an angle  $\varphi(\|y - f^{\pi_1}(z)\|^2)\eta$  in each symplectic two-dimensional subspace  $E_y^i \oplus E_y^{\hat{i}} = E_{f^{\pi_1}(z)}^i \oplus E_{f^{\pi_1}(z)}^{\hat{i}}$ , where  $\eta > 0$  is a degree of freedom to be considered in the sequel such that  $\eta \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . More precisely, if  $y \in B(f^{\pi_1}(z), \delta)$  we define  $S(y) : \mathbb{K}^{2\ell} \rightarrow \mathbb{K}^{2\ell}$  as the symplectic automorphism whose representation in the symplectic base  $\{e_i\}_i$  is given by

$$S(y) e_i = \cos(\eta \varphi(\|y - f^{\pi_1}(z)\|^2)) e_i - \sin(\eta \varphi(\|y - f^{\pi_1}(z)\|^2)) e_{\hat{i}}$$

and

$$S(y) e_{\hat{i}} = \sin(\eta \varphi(\|y - f^{\pi_1}(z)\|^2)) e_i + \cos(\eta \varphi(\|y - f^{\pi_1}(z)\|^2)) e_{\hat{i}}$$

for all  $i$ .

It is clear by the construction that the cocycle  $S$  and the developments in §3.2 that it is symplectic and coincides with the identity outside  $B(f^{\pi_1}(z), \delta)$ .

Moreover,  $S$  is a  $C^{r,\nu}$ -small perturbation of the identity cocycle. In fact, if  $\Delta(y) = \|y - f^{\pi_1}(z)\|^2$  then it follows from Faà di Bruno's formula that

$$\frac{\partial^n \varphi(\Delta(y))}{\partial y_k} = \sum \frac{n!}{m_1! 1!^{m_1} m_2! 2!^{m_2} \dots m_n! n!^{m_n}} \varphi^{(m_1 + \dots + m_n)}(\Delta(y)) \prod_{j=1}^n \left( \frac{\partial^j \Delta(y)}{\partial y_k} \right)^{m_j},$$



where the sum is over all vectors with nonnegative integers entries  $(m_1, \dots, m_n)$  such that we have  $\sum_{j=1}^n j \cdot m_j = n$ . It is not hard to check that  $\eta \frac{\partial^n \varphi(N(z))}{\partial y_k^n}$  is close to zero, and, in consequence, the cocycle  $B = A \circ S$  satisfies  $\|A - B\|_{r,\nu} \leq \varepsilon$ , provided that  $\eta$  is small enough. More precisely, if  $r = 0$  then  $\|A - B\|_{r,\nu}$  reduces to

$$\begin{aligned} \|A - B\|_\nu &= \sup_{x \neq y} \frac{\|(A - B)(x) - (A - B)(y)\|}{d(x, y)^\nu} \\ &= \sup_{x \neq y} \left\| \frac{A(x)[S(x) - id] - A(y)[S(y) - id]}{d(x, y)^\nu} \right\| \\ &= \sup_{x \neq y} \left\| \frac{[A(x) - A(y)]}{d(x, y)^\nu} [S(x) - id] - A(y) \frac{[S(y) - S(x)]}{d(x, y)} d(x, y)^{1-\nu} \right\| \\ &\leq \|A\|_{r,\nu} \|S(x) - id\| + \|A\| \|S(x)\|_{1,0} \text{diam}(M)^{1-\nu} \leq \varepsilon \end{aligned}$$

since  $\|S(y) - id\|$  and  $\|S(y)\|_{1,0}$  are arbitrarily small by choice of the constant  $\eta$ . For  $r \in \mathbb{N}$ , we have  $D^r(A \circ S)(y) = D^r A(S(y)) D^r S(y)$  and we can use the Cauchy-Schwarz inequality to deal with the  $C^r$  norm of the cocycles and estimate the Hölder constant of  $D^r(A \circ S)$  as above and we leave the details to the reader.

Moreover,  $B = A \circ S$  coincides with  $A$  outside of  $V$ . On the other hand, since  $H_{B,q,z}^u$  depends only on the values of the cocycle  $B$  on the points  $\{f^{-j}(q) : j \geq 0\} \cup \{f^{-j}(z) : j \geq 0\}$  which are outside of  $V$ , then  $H_{B,q,z}^u = H_{A,q,z}^u$ . Similarly, one has that  $H_{B,p,f^{2\pi_1}(z)}^s = H_{A,p,f^{2\pi_1}(z)}^s$  and, using the equality (4.3), we are reduced to check that

$$\begin{aligned} H_{B,p,z}^s(v_i) &= [B^{2\pi_1}(z)]^{-1} H_{B,p,f^{2\pi_1}(z)}^s(v_i) \\ &= [B^{2\pi_1}(z)]^{-1} (e_i) \end{aligned}$$

does not belong to any subspace generated by proper subsets of  $W$ . See Figure 2.

In fact, this holds because

$$\begin{aligned} B^{2\pi_1}(z) &= B(f^{2\pi_1-1}(z)) \circ \dots \circ B(f^{\pi_1+1}(z)) \circ [B(f^{\pi_1}(z))] \circ B(f^{\pi_1-1}(z)) \circ \dots \circ B(z) \\ &= A(f^{2\pi_1-1}(z)) \circ \dots \circ A(f^{\pi_1+1}(z)) \circ [A \circ S(f^{\pi_1}(z))] \circ A(f^{\pi_1-1}(z)) \circ \dots \circ A(z) \end{aligned}$$

and  $S$  induces a rotation of angle  $\eta$  in consecutive two-dimensional symplectic subspaces  $E_y^i \oplus E_y^{\hat{i}}$  on each symplectic two-dimensional subspace  $E_i \oplus E_{\hat{i}}$ , which are disjoint from the previous ones for some small  $\eta$ . Finally, since the holonomies  $B \mapsto H_{B,p,z}^s$  and  $B \mapsto H_{B,z,q}^u$  are differentiable in a neighborhood of  $A$  in the  $C^{r,\nu}$ -topology (cf. Lemma 2.9 in [38]) then it is clear that the previous property is an open condition. This finishes the proof of the lemma.  $\square$

Let us mention that the previous result will be of particular interest in §4.5 when the vectors  $\{w_i\}_{i=1}^{2\ell}$  are related to eigenvectors of periodic points with simple spectrum and obtained by means of unstable holonomies.

**4.5. Finishing the proof of Theorem A.** Fix  $f \in \text{Diff}^{1+\alpha}(M)$  and an  $f$ -invariant, ergodic, hyperbolic measure  $\mu$  with local product structure and  $\ell \geq 1$ . We prove that the set of cocycles

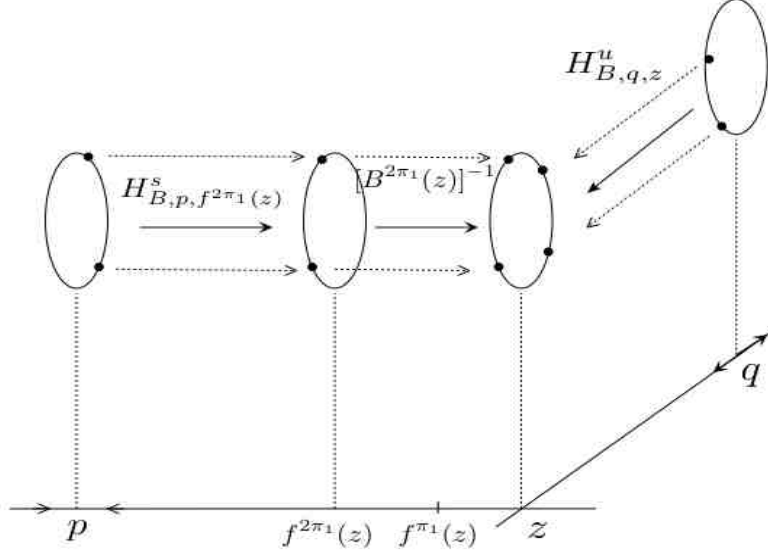


FIGURE 2. Stable and unstable holonomies.

in  $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  that have all zero Lyapunov exponents is contained in a compact set with empty interior and infinite codimension.

Let  $A \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  be such that  $\lambda^+(A, \mu) = 0$ , that is, so that  $F_A$  has only zero Lyapunov exponents and take an arbitrary  $\varepsilon > 0$  and also  $k \geq 2$ . It follows from Proposition 4.8 that there exists a holonomy block  $\mathcal{O}$  and distinct dominated periodic points  $\{p_i\}_{i=1}^k$  in  $\mathcal{O}$  and a cocycle  $\tilde{B} \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  satisfying  $\|A - \tilde{B}\|_{r,\nu} < \varepsilon/2$  and such that  $W_{\text{loc}}^u(p_i) \cap W_{\text{loc}}^s(p_{i+1}) \neq \emptyset$  consists of one point  $z_i$  and the Lyapunov spectrum of  $\tilde{B}^{\pi_i}(p_i)$  is real and simple, where  $\pi_i$  is the period of the periodic point  $p_i$  for all  $i = 1, \dots, k$ .

Let  $\{v_j^i : j = 1 \dots 2\ell\}$  be a symplectic base of eigenvectors for  $A^{\pi_i}(p_i)$  for each  $i = 1 \dots k$ . Consider also the symplectic base  $\{w_j^i = H_{\tilde{B}, p_i, z_i}^u(v_j^i) : j = 1 \dots 2\ell\}$  on the fiber of the heteroclinic point  $z_i$ , for  $i = 1 \dots k-1$ . Since the assertion of the Breaking Lemma (Lemma 4.9) is an open condition one can apply it recursively to each homoclinic point  $z_i$  to prove that there exists  $B \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  with  $\|B - \tilde{B}\|_{r,\nu} < \varepsilon/2$  and so that the unstable holonomies  $H_{B, p_i, z_i}^u = H_{\tilde{B}, p_i, z_i}^u$  coincide and  $H_{B, p_{i+1}, z_1}^s(v_j^i)$  does not belong to the subspace generated by any proper subset of  $W$ . Again, the later property is open in the  $C^{r,\nu}$ -topology and clearly implies

$$(h_{B, p_i, z_i}^u)_* m_{p_i} \neq (h_{B, p_{i+1}, z_1}^s)_* m_{p_{i+1}}, \quad \text{for all } i = 1 \dots k-1.$$

Together with Corollary 4.5 this implies that  $B$  has at least one non-zero Lyapunov exponent and proves that the set of cocycles in  $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$  with at least one non-zero Lyapunov exponent is an open and dense set. In addition, we also deduce that cocycles in  $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$

with all zero Lyapunov exponents are contained in codimension  $k$  topological submanifolds. Since  $k \geq 2$  was chosen arbitrary then it follows the second assertion and completes the proof of the theorem.

## 5. HAMILTONIAN LINEAR DIFFERENTIAL SYSTEMS OVER SUSPENSION FLOWS

In this section we prove Theorem B in the case of symplectic cocycles over non-uniformly hyperbolic suspension flows. Since the general case is more involving but uses some of these arguments, this intermediate step from discrete time to suspension flows will be useful to the reader. As a consequence we will deduce Corollary 2.7 on symplectic cocycles over non-uniformly hyperbolic flows. The strategy to deal with Hamiltonian linear differential systems over suspension flows is to make a reduction to the discrete time setting, in which case we consider an induced cocycle in the fiber that also depends on the roof function. It is here that we need to require the roof function to be bounded (recall Lemma 3.4). Moreover, an extra difficulty is caused since our perturbations are in the space of Hamiltonian linear differential system  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  as the infinitesimal generators of the fundamental solutions  $\Phi_H^t$  over the flow  $(X^t)_t$ . One of the main difficulties is really to analyze the variation of the holonomies for the cocycle  $\Psi_H$  as a function of the infinitesimal generators  $H$ .

So, given a compact Riemannian manifold  $M_0$ ,  $f : M_0 \rightarrow M_0$  a  $C^{1+\alpha}$  diffeomorphism endowed with an invariant and ergodic hyperbolic measure  $\mu$  with local product structure and  $\varrho : M_0 \rightarrow [0, +\infty)$  a Hölder continuous roof function which is bounded away from zero (without loss of generality assume the height is larger than one), we consider the corresponding suspension flow  $(X^t)_{t \geq 0}$  over  $f$  acting on the space

$$M = \{(x, t) \in M_0 \times \mathbb{R}_0^+ : 0 \leq t \leq \varrho(x)\} / \sim, \quad (5.1)$$

where  $(x, \varrho(x)) \sim (f(x), 0)$ , was described before in §3.3. It is clear from the definition that the  $(X^t)_t$ -invariant probability measure  $\hat{\mu} = (\mu \times \text{Leb}) / \int \varrho d\mu$  has the local product structure.

**5.1. A reduction to the base dynamics.** Our strategy to deal with suspension flows is to make a reduction of the dynamics, cocycle and invariant measures by an inducing process to corresponding objects over a non-uniformly hyperbolic map.

**5.1.1. A cocycle reduction to the base dynamics.** Our approach here is to use a reduction of the time-continuous Hamiltonian to the case of discrete time setting. For that purpose consider the cocycle  $\Psi_H : \Sigma \times \mathbb{K}^{2\ell} \rightarrow \Sigma \times \mathbb{K}^{2\ell}$  induced naturally from  $\Phi_H^t$  on  $\Sigma = M_0 \times \{0\}$  given by

$$\Psi_H(x, v) = \left( f(x), \Phi_H^{\varrho(x)}(x) v \right).$$

Given  $n \geq 1$  set

$$\Psi_H^n(x) = \Psi_H(f^{n-1}(x)) \circ \cdots \circ \Psi_H(f(x)) \circ \Psi_H(x),$$

and notice that

$$\Psi_H^n(x) = \Phi_H^{\varrho^{(n)}(x)}(x), \quad (5.2)$$

where  $\varrho^{(n)}$  is an inducing of the original roof function and defined in (3.4).

For simplicity reasons, as discussed previously, we shall assume that the roof function  $\varrho$  is Lipschitz and our first step, which is a generalization of Lemma 3.3, is to obtain the Lipschitz regularity for the induced cocycle.

**Lemma 5.1.** *The induced cocycle  $\Psi_H$  is Lipschitz continuous.*

*Proof.* Since we assume that  $H$  is Lipschitz, it follows from Lemma 3.3 that the time- $t$  cocycle  $\Phi_H^t$  is Lipschitz continuous for every  $t \geq 0$ . Hence, it follows from (5.2) that

$$\|\Psi_H(x) - \Psi_H(y)\| \leq \|\Phi_H^{\varrho(x)}(x) - \Phi_H^{\varrho(x)}(y)\| + \|\Phi_H^{\varrho(x)}(y) - \Phi_H^{\varrho(y)}(y)\|.$$

On the one hand, since  $\varrho$  is continuous and  $M_0$  is compact then  $\varrho$  is bounded from above by the constant  $\varrho_1$ , the first term in the right hand side is bounded by  $K(\varrho_1)d(x, y)$ , for some  $K(\varrho_1) > 0$ . On the other hand, since  $\Phi_H^0 = id$  and the roof function  $\varrho$  is Lipschitz, then the rightmost term above is bounded by

$$\begin{aligned} \|\Phi_H^{\varrho(x)}(y) - \Phi_H^{\varrho(y)}(y)\| &\leq \|\Phi_H^{\varrho(x)}(y)\| \|\Phi_H^0(y) - \Phi_H^{\varrho(y)-\varrho(x)}(y)\| \\ &\leq e^{K\varrho_1} C |\varrho(y) - \varrho(x)| \leq C' d(x, y) \end{aligned}$$

for some positive constants  $C, C'$ . □

In the remaining of this section we assume that  $H \in C^{r,\alpha}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  is such that all the Lyapunov exponents for  $\Phi_H^t$  with respect to  $\hat{\mu}$  are equal to zero. In particular, it follows from Lemma 3.4 that the discrete-time cocycle  $\Psi_H$  over  $(f, \mu)$  has only zero Lyapunov exponents (here we use  $f : \Sigma \rightarrow \Sigma$  by a small abuse of notation). So, proceeding as in §4.2, there exist constants  $K, \tau, N, \theta$  such that  $3\theta < \tau$  and the holonomy block  $\mathcal{O} = \mathcal{H}(K, \tau) \cap \mathcal{D}_{\Psi_H}(N, \theta) \subset \Sigma$ , for the cocycle  $\Psi_H$ , has positive  $\mu$ -measure. In consequence we obtain from Proposition 4.2 the existence of unstable holonomies. More precisely,

**Corollary 5.2.** *For every  $x \in \mathcal{O}$  and  $y \in W_{loc}^u(x) \subset \Sigma$ , there exists  $C_1 > 0$  and a symplectic linear transformation  $L_{H,x,y}^u : \{x\} \times P\mathbb{K}^{2\ell} \rightarrow \{y\} \times P\mathbb{K}^{2\ell}$  such that:*

- (1)  $L_{H,x,x}^u = id$  and  $L_{H,x,z}^u = L_{H,y,z}^u \circ L_{H,x,y}^u$ ;
- (2)  $\Psi_H(f^{-1}(y)) \circ L_{H,f^{-1}(x),f^{-1}(y)}^u \circ \Psi_H(x)^{-1} = L_{H,x,y}^u$ ;
- (3)  $\|L_{H,x,y}^u - id\| \leq C_1 d(x, y)$  and
- (4)  $L_{H,f^j(y),f^j(z)}^u = \Psi_H^j(z) \circ L_{H,y,z}^u \circ \Psi_H^j(y)^{-1}$  for all  $j \in \mathbb{Z}$

for every  $x, y, z$  in the same local unstable manifold.

Since the Poincaré return map  $f$  of the flow to the global cross-section  $\Sigma$  is such that  $\mu$  has local product structure, for any positive measure holonomy block  $\mathcal{O}$  and for all  $x \in \text{supp}(\mu|_{\mathcal{O}})$  let  $\mathcal{N}_x(\mathcal{O}, \delta)$ ,  $\mathcal{N}_x^u(\mathcal{O}, \delta)$  and  $\mathcal{N}_x^s(\mathcal{O}, \delta)$  be the induced neighborhoods of  $x$  in  $\mathcal{O}$  with local product structure defined similarly as before. Moreover, by some abuse of notation, when no confusion is possible on the Hamiltonian  $H$  we shall denote by  $h_{x,y}^s$  and  $h_{x,y}^u$  the projectivization of the stable and unstable holonomies, respectively.

5.1.2. *Invariant measures reduction to the base dynamics.* Here we show that one can also reduce invariant measures by the cocycle to the ones of the induced cocycle. Let  $(\varphi_H^t)_t$  denote the cocycle over  $(X^t)_t$  projectivized from  $(\Phi_H^t)_t$  and we will also denote by  $\psi_H$  the projectivized cocycle obtained from  $\Psi_H$ .

**Lemma 5.3.** *Let  $m$  be a  $(\varphi_H^t)_t$ -invariant probability measure such that  $\Pi_* m = \hat{\mu}$  and  $m = \int_M m_z d\hat{\mu}(z)$  be a disintegration of  $m$ . Then  $(\varphi_H^t(z))_* m_z = m_{X^t(z)}$  for  $\mu$ -almost every  $z \in \Sigma$  and all  $0 \leq t \leq \varrho(z)$ .*

*Proof.* Notice that, by construction,  $\{(x, \varrho(x)) : x \in M_0\} \subset M$  is a zero  $\hat{\mu}$ -measure set and, consequently,

$$\hat{\mu}(\{(x, t) \in M_0 \times \mathbb{R}^+ : 0 \leq t < \varrho(x)\}) = 1.$$

Moreover, the family  $(\{z\} \times [0, \varrho(z)))_{z \in \Sigma}$  defines a measurable partition of the previous set in the sense of Rokhlin. The same holds for the partition into points of each segment of orbit  $\{z\} \times [0, \varrho(z))$ . Therefore, given any  $(\varphi_H^t)_t$ -invariant probability measure  $m$  such that  $\Pi_* m = \hat{\mu}$  there exists a  $\hat{\mu}$ -almost everywhere defined family of probability measures  $(m_{X^t(z)})_{z \in \Sigma, t \in [0, \varrho(z))}$  such that  $\text{supp}(m_{X^t(z)}) \subset \{X^t(z)\} \times P\mathbb{K}^{2\ell}$  and  $m = \int m_{X^t(z)} d\hat{\mu}$ . More precisely, using  $\hat{\mu} = (\mu \times \text{Leb}) / \int \varrho d\mu$  one can write

$$m(E) = \frac{1}{\int \varrho d\mu} \int \left[ \int_0^{\varrho(z)} m_{X^t(z)}(E) dt \right] d\mu(z)$$

for all measurable sets  $E \subset M \times P\mathbb{K}^{2\ell}$ .

Now, by invariance of  $m$ , for all  $t \in \mathbb{R}$  we get that  $(\varphi_H^t)_* m = m$  and, since any two disintegrations of the same probability measure coincide almost everywhere we get  $m_{X^t(z)} = (\varphi_H^t(z))_* m_z$  for  $\mu$ -almost every  $z \in \Sigma$  and Lebesgue almost every  $t \in [0, \varrho(z))$ . Finally, just observe that one can consider on each fiber  $\{z\} \times [0, \varrho(z))$  for the disintegration the elements given by  $t \mapsto (\varphi_H^t(z))_* m_z$ , that vary continuously with  $t$  in the weak\* topology.  $\square$

**5.2. Continuous disintegration and criterion for non-zero Lyapunov exponents.** Here we just collect some of the previous ingredients and show that only zero Lyapunov exponents for the time-continuous cocycle  $\varphi_H^t$  over  $(X^t, \hat{\mu})$  implies on a rigid condition on the disintegration of some  $\psi_H$ -invariant probability measures. Throughout, let  $m$  be a  $(\varphi_H^t)_t$ -invariant probability measure such that  $\Pi_* m = \hat{\mu}$ .

**Lemma 5.4.** *The measure  $m$  is completely determined by a probability measure  $m_\Sigma$  on  $\Sigma \times P\mathbb{K}^{2\ell}$  such that  $(\psi_H)_* m_\Sigma = m_\Sigma$  and  $\Pi_* m_\Sigma = \mu$ .*

*Proof.* The Lemma 5.3 implies that  $m$  is completely determined by probability measures  $(m_z)_{z \in \Sigma}$ . Moreover, it is clear from the invariance condition that  $m_\Sigma = \int m_z d\mu$  is  $\psi_H$ -invariant, because we have  $(\varphi_H^t(z))_* m_z = m_{X^t(z)}$  for all  $0 \leq t < \varrho(z)$  and continuity in the weak\* topology.

On the one hand, since  $\Pi_* m = \hat{\mu}$ , for any measurable cylinder  $E = E_1 \times [0, b]$  contained in  $\{(z, t) : z \in M_0 \text{ and } 0 \leq t < \varrho(z)\}$  we get

$$(\Pi_* m)(E) = m(\pi^{-1}(E)) = \hat{\mu}(E) = \frac{1}{\int \varrho d\mu} \times \mu(E_1) \times \int_0^b dt.$$

On the other hand, using the invariance condition  $(\varphi_H^t(z))_* m_z = m_{X^t(z)}$  then one can write

$$m = \int_M m_z d\hat{\mu}(z) = \frac{1}{\int_{\mathcal{Q}} d\mu} \int_{M_0} \left[ \int_0^{\varrho(z)} m_{X^t(z)} dt \right] d\mu(z).$$

Therefore for the set  $E$  as above we get

$$(\Pi_* m)(E) = m(\pi^{-1}(E)) = \frac{1}{\int_{\mathcal{Q}} d\mu} \int_0^b m_{\Sigma}(\pi^{-1}(E) \cap \Sigma) dt = \frac{1}{\int_{\mathcal{Q}} d\mu} \Pi_* m_{\Sigma}(E) \times \int_0^b dt.$$

By continuity, it follows that  $(\varphi_H^{\varrho(z)})_* m_z = m_{f(z)}$  for  $\mu$ -almost every  $z$  and consequently we have  $(\psi_H(z))_* m_z = m_{f(z)}$  for  $\mu$ -almost every  $z$ . This shows that  $(\psi_H)_* m_{\Sigma} = m_{\Sigma}$  and  $\Pi_* m_{\Sigma} = \mu$ , finishing the proof of the lemma.  $\square$

This put us in a position to make use of the machinery of Section 4. Proposition 4.3 applied to the cocycle  $\psi_H$  yields the following immediate consequence (a similar result holds for unstable holonomies):

**Corollary 5.5.** *Let  $\mathcal{O}$  be a positive  $\mu$ -measure holonomy block, consider  $x \in \text{supp}(\mu \mid \mathcal{O})$  and set the neighborhoods  $\mathcal{N}_x(\mathcal{O}, \delta)$  of  $x$  as above. Then, every  $\psi_H$ -invariant probability measure  $m_{\Sigma}$  with  $\Pi_* m_{\Sigma} = \mu$  admits a continuous disintegration on  $\text{supp}(\mu \mid \mathcal{N}_x(\mathcal{O}, \delta))$ . Moreover,*

$$m_z = (h_{H,y,z}^s)_* m_y \quad \text{and} \quad m_z = (h_{H,w,z}^u)_* m_w \quad (5.3)$$

for all  $y, z, w \in \text{supp}(\mu \mid \mathcal{N}_x(\mathcal{O}, \delta))$  such that  $y, z$  belong to the same strong-stable local manifold and  $z, w$  belong to the same strong-unstable local manifold.

Let  $\mathcal{O}$  be a positive  $\mu$ -measure holonomy block for a cocycle  $\Psi_H$  and let  $x \in \text{supp}(\mu \mid \mathcal{O})$  be as above. By continuity of the disintegration and Lemma 4.4, if  $m_{\Sigma}$  is an  $\psi_H$ -invariant probability measure such that  $\Pi_* m_{\Sigma} = \mu$  and  $p \in \text{supp}(\mu \mid \mathcal{N}_x(\mathcal{O}, \delta))$  is a periodic point of period  $\pi$  for  $f$  then  $\Psi_H^{\pi}(p)_* m_p = m_p$  and, in addition, if  $\Psi_H^{\pi}(p)$  has all real and distinct eigenvalues, then there exist elements  $\{v_i\}_{i=1 \dots 2\ell}$  in  $P\mathbb{K}^{2\ell}$  and a probability vector  $\{\alpha_i\}_{i=1 \dots 2\ell}$  such that  $m_p = \sum_{i=1}^{\ell} \alpha_i \delta_{v_i}$ . Moreover, as in Corollary 4.5, given  $p, q \in \text{supp}(\mu \mid \mathcal{N}_x(\mathcal{O}, \delta))$  dominated periodic points for  $f$  and  $z$  is the unique point in the heteroclinic intersection  $W_{\text{loc}}^u(q) \cap W_{\text{loc}}^s(p)$  then

$$m_z = (h_{p,z}^u)_* m_p = (h_{q,z}^s)_* m_q.$$

Recall that if  $p \in \Sigma$  is a dominated periodic point of period  $\pi$  for  $f$ , then  $h_{p,z}^s$  is the projectivization of the stable holonomy for the cocycle  $\Psi_H$  over  $f$ , which is given by

$$L_{H,p,z}^s = \lim_{n \rightarrow \infty} \Psi_H^{\pi n}(z)^{-1} \Psi_H^{\pi n}(p). \quad (5.4)$$

### 5.3. Realization of symplectic time- $t$ actions by Hamiltonian linear differential systems.

In the present section we show that for any given Hamiltonian linear differential system and any small perturbation of the symplectomorphism given by the time-one map of its solution, there exists a Hamiltonian linear differential system close to the original one which realizes the perturbation map. This is the content of Lemma 5.6 and we point out that it is done for any general flow not necessarily a suspension one. Let us now prepare the set up.

Given  $S \in sp(2\ell, \mathbb{K})$ , we can view  $S$  as the time-one map of the linear flow solution of the linear variational equation  $\dot{u}(t) = \mathbf{S}(t) \cdot u(t)$  with initial condition  $u(0) = id$ . In other words,  $u(t) = \Phi_S^t$  is solution of  $\dot{u}(t) = \mathbf{S}(t) \cdot u(t)$ , and  $\Phi_S^1 = S$ . Since, by Gronwall's inequality we have

$$\|S_t\|_{r,v} \leq \exp \left\{ \int_0^t \|\mathbf{S}(s)\|_{r,v} ds \right\}, \forall t \geq 0 \quad (5.5)$$

we say that  $S \in sp(2\ell, \mathbb{K})$  is  $\delta$ - $C^{r,v}$ -close to identity if  $\mathbf{S}$  is  $\delta$ - $C^{r,v}$ -small, i.e.,  $\|\mathbf{S}\|_{r,v} < \delta$ .

**Lemma 5.6.** *Let be given  $H \in C^{r,v}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  over a flow  $X^t: M \rightarrow M$ , any nonperiodic point  $x \in M$  (or periodic with period  $> 1$ ) and  $\varepsilon > 0$ . There exists  $\delta = \delta(H, \varepsilon) > 0$  such that if  $S \in sp(2\ell, \mathbb{K})$  is isotopic to the identity and  $\delta$ - $C^{r,v}$ -close to  $id$ , then there exists  $H_0 \in C^{r,v}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  satisfying:*

- (a)  $\|H_0 - H\|_{r,v} < \varepsilon$  and
- (b)  $\Phi_{H_0}^1(x) = \Phi_H^1(x) \circ S$ .

*Proof.* By the tubular flowbox theorem (see [1, 34]) there exists a smooth change of coordinates so that there exists a local conjugacy of  $X$  on a neighborhood of the segment of orbit  $\{X^t(x): t \in [0, 1]\}$  to a constant vector field on  $\mathbb{R}^d$  where  $d = 2\ell = \dim(M)$ . With this assumption we consider  $x = \vec{0}$  and

$$\{X^t(x): t \in [0, 1]\} = \{(t, 0, \dots, 0) \in \mathbb{R}^d: t \in [0, 1]\} \subset \frac{\partial}{\partial x_1},$$

where  $\frac{\partial}{\partial x_1}$  denotes the direction spanned by direction  $x_1 = (1, 0, \dots, 0)$ . Given  $\rho > 0$  let  $B(\vec{0}, \rho) \subset \left(\frac{\partial}{\partial x_1}\right)^\perp$  denotes the ball centered in  $\vec{0}$  of radius  $\rho$  contained in the hyperplane orthogonal to  $\frac{\partial}{\partial x_1}$ . Our perturbation will be performed inside the cylinder

$$C = B(\vec{0}, \rho) \times [0, 1] = \{X^t(B(\vec{0}, \rho)): t \in [0, 1]\}.$$

Using the fact that  $M$  is compact we can take

$$K := \max_{z \in M, t \in [0, 1]} \{\|\Phi_H^t(z)\|_{r,v}, \|(\Phi_H^t(z))^{-1}\|_{r,v}, \|H\|_{r,v}\}. \quad (5.6)$$

Fix any  $\varepsilon > 0$  and choose

$$\delta := \frac{\varepsilon}{6K^3}.$$

Now, consider an isotopy  $S_t \in sp(2\ell, \mathbb{K})$ , for every  $t \in [0, 1]$ , such that:

- (1)  $S_0 = id$  and  $S_1 = S$ ;
- (2)  $S_t$  is the solution of the linear variational equation  $\dot{u}(t) = \mathbf{S}(t) \cdot u(t)$  with infinitesimal generator  $\mathbf{S}$  satisfying the inequality

$$\|\mathbf{S}\|_{r,v} := \sup_{0 \leq j \leq r} \|D^j \mathbf{S}(t)\| + \sup_{t \neq s} \frac{\|\mathbf{S}(t) - \mathbf{S}(s)\|}{|t - s|^\nu} < \delta.$$

Consider the  $C^\infty$  bump-function  $\alpha: [0, \infty[ \rightarrow [0, 1]$ , with  $\alpha(s) = 0$  if  $s \geq \rho$  and  $\alpha(s) = 1$  if  $s \in [0, \rho/2]$ . Given  $z \in B(\vec{0}, \rho)$  we consider an isotopy  $S_t(z) \in sp(2\ell, \mathbb{K})$  for every  $t \in [0, 1]$  such that:

- (I)  $S_0(z) = id$  and  $S_1(z) = \alpha(\|z\|^2)S$ ;  
 (II)  $S_t(z)$  is the solution of  $\partial_t u(t, z) = \mathbf{S}(t, z) \cdot u(t, z)$  with infinitesimal generator  $\mathbf{S}$  satisfying the inequality

$$\|\mathbf{S}(t, z)\|_{r, \nu} := \sup_{0 \leq j \leq r} \sup_{t \in [0, 1]} \|D^j \mathbf{S}(z + (t, 0, \dots, 0))\| + \sup_{x \neq y} \frac{\|\mathbf{S}(x) - \mathbf{S}(y)\|}{d(x, y)^\nu} < \delta.$$

Let  $\Upsilon_t(z) = \Phi'_H(z)\alpha(\|z\|)S_t(z)$  and we consider time derivatives:

$$\begin{aligned} \Upsilon_t(z)' &= \Phi'_H(z)' \alpha(\|z\|) S_t(z) + \Phi'_H(z) (\alpha(\|z\|) S_t(z))' \\ &= H(X^t(z)) \Phi'_H(z) \alpha(\|z\|) S_t(z) + \Phi'_H(z) (\alpha(\|z\|) S_t(z))' \\ &= H(X^t(z)) \Upsilon_t(z) + \Phi'_H(z) (\alpha(\|z\|) S_t(z))' (\Upsilon_t(z))^{-1} \Upsilon_t(z) \\ &= [H(X^t(z)) + P(X^t(z))] \cdot \Upsilon_t(z). \end{aligned}$$

Define, for  $t \in [0, 1]$  and  $z \in B(\vec{0}, \rho)$  the perturbation

$$H_0(X^t(z)) = H(X^t(z)) + P(X^t(z)), \quad (5.7)$$

where

$$\begin{aligned} P(X^t(z)) &= \Phi'_H(z) (\alpha(\|z\|) S_t(z))' (\Upsilon_t(z))^{-1} \\ &= \Phi'_H(z) \alpha(\|z\|) S_t(z)' (\Phi'_H(z) \alpha(\|z\|) S_t(z))^{-1} \\ &= \Phi'_H(z) \alpha(\|z\|) S_t'(z) (S_t(z))^{-1} \alpha(\|z\|)^{-1} (\Phi'_H(z))^{-1} \\ &= \Phi'_H(z) S_t'(z) (S_t(z))^{-1} (\Phi'_H(z))^{-1}. \end{aligned}$$

Thus, in the flowbox coordinates  $(z, t) \in \mathcal{C}$ , where recall  $z \in B(\vec{0}, \rho)$  and  $t \in [0, 1]$ , we have:

$$P(X^t(z)) = \Phi'_H(z) \mathbf{S}(t, z) (\Phi'_H(z))^{-1},$$

and outside the flowbox cylinder  $\mathcal{C}$  we let  $P = [0]$ .

First we claim that, for all  $t$  and  $z$ , we have  $P(X^t(z)) \in \mathfrak{sp}(2\ell, \mathbb{R})$ , or equivalently that  $\mathbf{S} := (S_t)'(S_t)^{-1} \in \mathfrak{sp}(2\ell, \mathbb{R})$ . Then, let us prove that  $J\mathbf{S} + \mathbf{S}^T J = 0$ ; for any  $S \in \mathfrak{sp}(2\ell, \mathbb{R})$  we have the symplectic identities:  $J^{-1} = J^T = -J$ ,  $S^T J S = J$  and  $S^{-1} = J^{-1} S^T J$ .

$$\begin{aligned} J\mathbf{S} + \mathbf{S}^T J &= J(S_t)'(S_t)^{-1} + [(S_t)'(S_t)^{-1}]^T J \\ &= J(S_t)'(S_t)^{-1} + ((S_t)^{-1})^T ((S_t)')^T J \\ &= J[-(S_t)'(S_t)^{-1} J - J((S_t)^{-1})^T ((S_t)')^T] J \\ &= J[(S_t)' J^{-1} (S_t)^T + S_t J^{-1} (S_t')^T] J \\ &= J[S_t J^{-1} (S_t)^T]' J \\ &= J[J^{-1}]' J = 0. \end{aligned}$$

Second, we will prove condition (a) of the conclusions of the lemma, i.e., that  $\|H_0 - H\|_{r, \nu} < \varepsilon$  or, equivalently, that  $\|P\|_{r, \nu} < \varepsilon$ . We will perform the computations for  $r = 0$  with all the details. For  $r \in \mathbb{N}$  we can estimate easily using the chain rule and Cauchy-Schwarz inequality. Whenever we consider points  $x, y$  in the tubular flowbox  $\mathcal{C}$  (the support of the perturbation) we write them in the flowbox coordinates  $x = (z, t)$ ,  $y = (w, s)$ , where  $t, s \in [0, 1]$  and  $z, w \in$



$B(\vec{0}, \rho)$ . Then, the estimates on  $\|P\|_{0,\nu} = \|P\|_\nu$  can be reduced to the estimates on the product structure  $B(\vec{0}, \rho) \times [0, 1]$  by using a triangular argument.

- For  $z_t, w_t$  inside the same laminar section in  $C$ , i.e.,  $z_t = (z, t)$  and  $w_t = (w, t)$ ;

$$\begin{aligned}
\|P\|_\nu &= \sup_{z_t \neq w_t} \frac{\|P(z_t) - P(w_t)\|}{d(z_t, w_t)^\nu} \\
&= \sup_{z \neq w} \frac{\|\Phi_H^t(z) \mathbf{S}(t, z) (\Phi_H^t(z))^{-1} - \Phi_H^t(w) \mathbf{S}(t, w) (\Phi_H^t(w))^{-1}\|}{d(z, w)^\nu} \\
&\leq \sup_{z \neq w} \frac{\|\Phi_H^t(z) [\mathbf{S}(t, z) - \mathbf{S}(t, w)] (\Phi_H^t(z))^{-1}\|}{d(z, w)^\nu} + \frac{\|[\Phi_H^t(z) - \Phi_H^t(w)] \mathbf{S}(t, w) (\Phi_H^t(z))^{-1}\|}{d(z, w)^\nu} \\
&\quad + \frac{\|\Phi_H^t(w) \mathbf{S}(t, w) [(\Phi_H^t(z))^{-1} - (\Phi_H^t(w))^{-1}]\|}{d(z, w)^\nu} \\
&\stackrel{(5.6)}{\leq} \sup_{z \neq w} \frac{K^2 \|\mathbf{S}(t, z) - \mathbf{S}(t, w)\|}{d(z, w)^\nu} + \frac{K \|\Phi_H^t(z) - \Phi_H^t(w)\| \|\mathbf{S}(t, w)\|}{d(z, w)^\nu} \\
&\quad + \frac{K \|\mathbf{S}(t, w)\| \|(\Phi_H^t(z))^{-1} - (\Phi_H^t(w))^{-1}\|}{d(z, w)^\nu} \\
&\leq K^2 \delta + 2Ke^\delta \delta < \varepsilon.
\end{aligned}$$

- For  $z_t, z_s$  inside the same orbit in  $C$ ;

$$\begin{aligned}
\|P\|_\nu &= \sup_{z_t \neq z_s} \frac{\|P(z_t) - P(z_s)\|}{d(z_t, z_s)^\nu} \\
&= \sup_{t \neq s} \frac{\|\Phi_H^t(z) \mathbf{S}(t, z) (\Phi_H^t(z))^{-1} - \Phi_H^s(z) \mathbf{S}(s, z) (\Phi_H^s(z))^{-1}\|}{|t - s|^\nu} \\
&\leq \sup_{t \neq s} \frac{\|\Phi_H^t(z) [\mathbf{S}(t, z) - \mathbf{S}(s, z)] (\Phi_H^t(z))^{-1}\|}{|t - s|^\nu} + \frac{\|[\Phi_H^t(z) - \Phi_H^s(z)] \mathbf{S}(s, z) (\Phi_H^t(z))^{-1}\|}{|t - s|^\nu} \\
&\quad + \frac{\|\Phi_H^s(z) \mathbf{S}(s, z) [(\Phi_H^t(z))^{-1} - (\Phi_H^s(z))^{-1}]\|}{|t - s|^\nu} \\
&\leq \sup_{t \neq s} \frac{K^2 \|\mathbf{S}(t, z) - \mathbf{S}(s, z)\|}{|t - s|^\nu} + \frac{K \|\Phi_H^t(z) (id - \Phi_H^{s-t}[X^t(z)])\| \|\mathbf{S}(t, w)\|}{|t - s|^\nu} \\
&\quad + \frac{K \|\mathbf{S}(t, w)\| \|(\Phi_H^t(z))^{-1} (id - (\Phi_H^{s-t}(X^t(z)))^{-1})\|}{|t - s|^\nu} \\
&\leq \sup_{t \neq s} K^2 \delta + K^2 \delta \frac{\|id - \Phi_H^{s-t}(X^t(z))\|}{|t - s|^\nu} + K^2 \delta \frac{\|id - (\Phi_H^{s-t}(X^t(z)))^{-1}\|}{|t - s|^\nu} \\
&\leq K^2 \delta + \sup_{t \neq s} K^2 \delta \left( \frac{\|id - \Phi_H^{s-t}(X^t(z))\|}{|t - s|} + \frac{\|id - (\Phi_H^{s-t}(X^t(z)))^{-1}\|}{|t - s|} \right) |t - s|^{1-\nu} \\
&\leq K^2 \delta + \sup_{t \neq s} K^2 \delta (2\|H\|) |t - s|^{1-\nu} \\
&\leq K^2 \delta + 2K^3 \delta |t - s|^{1-\nu} \leq 3K^3 \delta < \varepsilon.
\end{aligned}$$

Finally, we will prove condition (b) of the conclusions of the lemma, i.e., that we have the equality  $\Phi_{H_0}^1(x) = \Phi_H^1(x) \circ S$ . We are considering  $x = \vec{0}$  so let us prove that  $\Phi_{H_0}^1(\vec{0}) = \Phi_H^1(\vec{0})S$ . Just observe that  $\Upsilon_t(z)$  is a solution of the linear differential equation

$$\dot{u}(t, z) = [H(X^t(z)) + P(X^t(z))] \cdot u(t, z) = H_0(X^t(z)) \cdot u(t, z). \quad (5.8)$$

But, given the initial condition  $u(0, z) = z$ , this solution is unique, say  $\Phi_{H_0}^t(\vec{0})$ . Since  $\Upsilon_t(z) = \Phi_H^t(z) \alpha(\|z\|) S_t(z)$  and it also satisfies (5.8) we obtain that  $\Phi_{H_0}^t(\vec{0}) = \Phi_H^t(z) \alpha(\|z\|) S_t(z)$ . Thus, for  $z = \vec{0}$  we get

$$\Phi_{H_0}^1(\vec{0}) = \Phi_H^1(\vec{0}) \alpha(\|z\|) S_1(\vec{0}) = \Phi_H^1(\vec{0}) \alpha(0) S_1(\vec{0}) = \Phi_H^1(\vec{0}) S,$$

and the lemma is proved.  $\square$

**5.4. Proof of Theorem B for suspension flows.** This subsection is devoted to the proof of Theorem B on the existence of non-zero Lyapunov exponents for Hamiltonian linear differential systems over suspension flows. The first part is constituted by some important perturbation arguments.

Let  $(X^t)_t$  be a suspension flow of a  $C^{1+\alpha}$  diffeomorphism  $f : \Sigma \rightarrow \Sigma$  on a Riemannian manifold  $\Sigma = M_0$  with Lipschitz continuous roof function  $\varrho$ . Recall that the case of Hölder roof

function can be dealt by changing the metric. Assume that  $(f, \mu)$  has local product structure and consider the  $(X^t)_t$ -invariant probability measure  $\hat{\mu} = (\mu \times \text{Leb}) / \int \varrho d\mu$  with local product structure with respect to the flow.

Let  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  be a Hamiltonian linear differential equation over the suspension flow  $(X^t)_t$  such that  $\lambda^+(H, \mu) = 0$ , that is, so that  $\Phi_H^t$  (hence also  $\Psi_H$ ) has only zero Lyapunov exponents at  $\hat{\mu}$ -almost everywhere. Take an arbitrary  $\varepsilon > 0$  and also  $k \geq 2$ . It is also a consequence of Proposition 4.8, the perturbation Lemma 5.6 and the boundeness of  $\varrho$  that there exists a holonomy block  $\mathcal{O} \subset \Sigma$  and exist distinct dominated periodic points  $\{p_i\}_{i=1}^k$  by  $f$  in the set  $\mathcal{O}$  and a Hamiltonian linear differential system  $\tilde{H}_0 \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  satisfying  $\|H - \tilde{H}_0\|_{r,\nu} < \varepsilon/2$  and such that  $W_{\text{loc}}^u(p_i) \cap W_{\text{loc}}^s(p_{i+1}) \neq \emptyset$  consists of one point  $z_i$  and the Lyapunov spectrum of

$$\Psi_{\tilde{H}_0}^{\pi_i}(p_i) = \Psi_{\tilde{H}_0}(f^{\pi_i-1}(p_i)) \cdots \circ \Psi_{\tilde{H}_0}(f(p_i)) \circ \Psi_{\tilde{H}_0}(p_i)$$

is real and simple, where  $\pi_i$  is the period of the periodic point  $p_i$  for all  $i = 1, \dots, k$ . We point out that these perturbations are performed in the linear differential system and intend to realize the same scheme constructed in the discrete case. For example, Lemma 4.7 is implemented in the following way: first we assume that we have a period point  $p_i$  for the base dynamics  $f$  on  $\Sigma$  with large period  $\pi_i$ . Then, we consider small neighborhoods of  $p_i$ ,  $U_i \subset \Sigma$  and, inside the thin tubular flowbox atop  $U_i$  defined by

$$U_{p_i}^\tau := \{X^t(x) \subset M : x \in U_i \text{ and } t \in [0, \varrho(x)]\},$$

which we assume to have height larger than 1, we will perform the perturbations of the linear differential system  $H$ . Take the collection

$$\{\Psi_H(f^{\pi_1-1}(p_i)), \dots, \Psi_H(f(p_i)), \Psi_H(p_i)\}$$

which can be rewritten as

$$\{\Phi_H^{\varrho(f^{\pi_1-1}(p_i))}(f^{\pi_1-1}(p_i)), \dots, \Phi_H^{\varrho(f(p_i))}(f(p_i)), \Phi_H^{\varrho(p_i)}(p_i)\}.$$

Now, we use Lemma 5.6 to input some rotational behavior on each  $\Psi_H(f^k(p_i))$  for  $k = 0, \dots, \pi_1 - 1$ . To be precise, we construct  $H_k$ , supported in  $U_{f^k(p_i)}^\tau$ , such that  $\|H_k - H\|_{r,\nu} < \varepsilon$  and

$$\Psi_{H_k}(f^k(p_i)) = \Phi_{H_k}^{\varrho(f^k(p_i))}(f^k(p_i)) = \Phi_H^{\varrho(f^k(p_i))}(f^k(p_i)) \circ R = \Psi_H(f^k(p_i)) \circ R,$$

where  $R$  play the part of the rotations in Lemma 4.7. Since all these sets  $U_k^\tau$  are pairwise disjoint we define  $H_0$  to be equal to  $H_k$  inside  $U_k^\tau$  and  $H_0 = H$  in  $M \setminus \bigcup_{k=0}^{\pi_1-1} U_{f^k(p_i)}^\tau$  where  $M$  is the whole castle over  $\Sigma$  as defined in (5.1) in which the suspension flow evolves. Notice that, since the roof function  $\varrho$  is bounded, the constant  $K$  considered in (5.6) is also uniformly bounded for  $t \in [0, \max(\varrho)]$ . Furthermore, the tubular flowbox considered in Lemma 5.6 for the support of the perturbation is already straightened out because the flow is a suspension.

In order to go on with the proof we need to “break the holonomy” by means of a small perturbation supported in the  $k$  heteroclinic intersections. In the vein of Lemma 4.9, we will show what to do in each one of the intersections. The following details should be taken into account: on one hand the holonomy properties described in Corollary 5.2, and to be used in the sequel, are with respect to  $\Psi_H$  which is a well-behaved cocycle (cf. Lemma 5.1), but on

the other hand the perturbation is on the linear differential system  $H$  and not in the (discrete) cocycle  $\Psi_H$  similarly of what we did above. We let  $W = \{w_i : i = 1 \dots 2\ell\}$  be any linearly independent set of vectors in the fiber  $P\mathbb{K}^{2\ell}$  over  $z \in W_{loc}^s(p) \cap W_{loc}^u(q)$  where  $p, q \in \Sigma$  are periodic  $f$  orbits. Given  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  and a symplectic base  $\{v_i : i = 1 \dots 2\ell\}$  in the fiber  $P\mathbb{K}^{2\ell}$  over  $p$ , there exists a Hamiltonian linear differential system  $H_0 \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  such that  $\|H - H_0\|_{r,\nu} < \varepsilon$ , the unstable holonomies coincide  $L_{H_0,q,z}^u = L_{H,q,z}^u$  and  $L_{H_0,p,z}^s(v_i)$  does not belong to the 1-dimensional subspace generated by  $w_j$  for all  $j$ . Moreover, the later property is open in the  $C^{r,\nu}$ -topology. Let us see succinctly how to define  $H_0$ .

First, we take a thin flowbox  $U_{f^{\pi_1}(z)}^\tau$ , where,  $z \in \Sigma$ ,  $U$  is a neighborhood of  $f^{\pi_1}(z)$  in  $\Sigma$  and  $\pi_1$  the  $f$ -period of  $p$ .

Second, using Lemma 5.6 we perform a perturbation supported in  $U_{f^{\pi_1}(z)}^\tau$  to spoil the strong accuracy of sending eigendirections into eigendirections by the holonomy as described in (5.3) of Corollary 5.5 which we remind in (5.9). Recall that  $\psi_H$  is the projectivized cocycle obtained from  $\Psi_H$ . By Corollary 5.5 we get that, under our context, every  $\psi_H$ -invariant probability measure  $m_\Sigma$  with  $\Pi_* m_\Sigma = \mu$  admits a continuous disintegration. Moreover, denoting by  $h_{H,p,z}^s$  the projectivization of the stable holonomy for the cocycle  $\Psi_H$  over  $f$  and by  $h_{H,q,z}^u$  the projectivization of the unstable holonomy for the cocycle  $\Psi_H$  over  $f$  we get

$$m_z = (h_{H,p,z}^s)_* m_p \quad \text{and} \quad m_z = (h_{H,q,z}^u)_* m_q \quad (5.9)$$

for some points  $q, z, p$  such that  $p, z$  belong to the same strong-stable local manifold and  $z, q$  belong to the same strong-unstable local manifold.

Now, as in the discrete case we intend to deal with  $k$  periodic orbits  $p_i$  and perform  $k$  disjoint supported perturbations  $H_i$  which are different from  $H$  exactly in  $U_{f^{\pi_i}(z_i)}^\tau$ , where  $\pi_i$  is the period of  $p_i$  and  $z_i$  its associated heteroclinic orbit. We observe that once we let  $H_i = H$  outside  $U_{f^{\pi_i}(z_i)}^\tau$  and noting that  $z_i$  is homoclinic with  $p_i$  and  $q_i$  there is no way for us to interfere on the unstable holonomy, thus  $L_{H_i,q_i,z_i}^u = L_{H,q_i,z_i}^u$  for all  $i = 1, \dots, k$ . One also has that  $L_{H_i,p_i,f^{2\pi_i}(z_i)}^s = L_{H,p_i,f^{2\pi_i}(z_i)}^s$ . Actually, we can be more precise because, since the perturbation is performed in  $U_{f^{\pi_i}(z_i)}^\tau$ , we have  $L_{H_i,p_i,f^{\pi_i+1}(z_i)}^s = L_{H,p_i,f^{\pi_i+1}(z_i)}^s$ . Now, using (5.4) we obtain

$$L_{H_i,p_i,z_i}^s = \lim_{n \rightarrow \infty} \Psi_H^{\pi_i n}(z_i)^{-1} \Psi_H^{\pi_i n}(p_i) = [\Psi_{H_i}^{2\pi_i}(z_i)]^{-1} L_{H,p_i,f^{2\pi_i}(z_i)}^s. \quad (5.10)$$

Notice that the sets  $U_{f^{\pi_i}(z_i)}^\tau$  are pairwise disjoint, so we define  $H_0$  to be equal to  $H_i$  inside  $U_{f^{\pi_i}(z_i)}^\tau$  and  $H_i = H$  in  $M \setminus \bigcup_{i=1}^k U_{f^{\pi_i}(z_i)}^\tau$ . Furthermore, each Hamiltonian linear differential system  $H_i$  should be such that

$$L_{H_i,p_i,z_i}^s(v_i) = [\Psi_{H_i}^{2\pi_i}(z_i)]^{-1} L_{H,p_i,f^{2\pi_i}(z_i)}^s(v_i) = [\Psi_{H_i}^{2\pi_i}(z_i)]^{-1}(e_i),$$

does not belong to any subspace generated by proper subsets of  $W$ .

Finally, the perturbation  $H_0$  will be such that for all  $i$  we have

$$(h_{H_0,p_i,z_i}^u)_* m_{p_i} \neq (h_{H_0,p_{i+1},z_i}^s)_* m_{p_{i+1}}.$$

Together with Corollary 5.5 this implies that  $\Psi_{H_0}$  has at least one non-zero Lyapunov exponent and proves that the set of linear differential systems in  $C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  over suspension flows

with bounded roof function and with at least one non-zero Lyapunov exponent is an open and dense set.

## 6. HAMILTONIAN LINEAR DIFFERENTIAL SYSTEMS: GENERAL CASE

In this section we prove Theorem B in the case of Hamiltonian skew-product flows over general non-uniformly hyperbolic flows. Some of the main differences with the case of suspension flows is that typically strong stable and unstable foliations are not jointly integrable and, consequently, one cannot a priori build global cross sections and apply directly the results concerning holonomy invariance for discrete time maps. In fact, not only the construction of good return time map is also more involving as one needs to prove that good hyperbolicity properties are inherited by projection of the local dynamics to the local cross section.

The strategy here is to prove that nonuniform hyperbolicity for the flow yields some non-uniform hyperbolicity of the Poincaré first return map to some local smooth cross-section (recall Lemma 2.3). Then, one induced a discrete time cocycle and reproduce the ideas from the suspension flow setting back in Section 5. The key arguments are to be able to define properly an invertible return map with some nonuniform hyperbolicity and keeping track of a local product structure.

As before we endow  $C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  with the  $C^{r,\nu}$ -topology defined using the norm

$$\|H\|_{r,\nu} = \sup_{0 \leq j \leq r} \sup_{x \in M} \|D^j H(x)\| + \sup_{x \neq y} \frac{\|H(x) - H(y)\|}{\|x - y\|^\nu},$$

where  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  and  $x, y \in M$ .

### 6.1. Non-uniform hyperbolicity for the flow and hyperbolicity for local Poincaré maps.

Let us consider a smooth flow  $X^t: M \rightarrow M$  preserving a hyperbolic and ergodic probability measure  $\hat{\mu}$  with local product structure as in Definition 2.4. By Pesin theory, there exists a  $\hat{\mu}$ -full measure set  $\mathcal{P}$  and measurable functions  $K: \mathcal{P} \rightarrow (0, +\infty)$  and  $\tau: \mathcal{P} \rightarrow (0, +\infty)$  so that for any given  $x \in \mathcal{P}$  there is a well defined local stable manifold  $W_{loc}^s(x)$  such that  $T_x W_{loc}^s(x) = E_x^s$  and  $d(X^t(y), X^t(z)) \leq K_x e^{-\tau_x t} d(y, z)$ , for every  $y, z \in W_{loc}^s(x)$  and  $t \geq 0$ . Similar property holds for local unstable manifolds.

Therefore, given positive constants  $K, \tau$  the points in the hyperbolic block  $\mathcal{H}(K, \tau)$  are such that both the local invariant manifolds  $W_{loc}^s(x)$  and  $W_{loc}^u(x)$  have uniform size, uniform contraction on  $W_{loc}^s(x)$ , uniform backward contraction on  $W_{loc}^u(x)$  and vary continuously with  $x \in \mathcal{H}(K, \tau)$ . In particular, if  $X$  denotes the vector field associated to  $(X^t)_t$ , i.e.  $X(x) := \frac{d}{dt} X^t(x)|_{t=0}$ , then the angle between the vector  $X(x)$  and any of the subspaces  $E_x^s$  or  $E_x^u$  varies continuously in  $\mathcal{H}(K, \tau)$  and consequently is bounded away from zero on the hyperbolic block.

We proceed to build some projective hyperbolicity on some transversal cross-section to the flow.

**Lemma 6.1.** *Let  $\Lambda$  be a positive  $\hat{\mu}$ -measure subset for the flow  $(X^t)_t$ . Given a regular point  $x \in \text{supp}(\hat{\mu} \upharpoonright_\Lambda)$  there exists a smooth local cross section  $\Sigma$  to the flow at  $x$  and a tubular neighborhood  $U_x^\delta$  of  $x$  such that  $\hat{\mu}$ -almost every  $y \in U_x^\delta$  has infinitely many returns to  $\Lambda \cap U_x^\delta$ .*

*Proof.* Let  $x \in \text{supp}(\hat{\mu} \mid_{\Lambda})$  for some set  $\Lambda$  such that  $\hat{\mu}(\Lambda) > 0$  and  $\hat{\mu}$ -invariant with respect to  $X^t$ , then for any open set  $V_x$  containing  $x$  we have  $\hat{\mu}(V_x) > 0$ . Since  $x$  is regular we can consider, in local charts, a small cross section  $\Sigma$  to  $X(x)$ , say normal to  $X(x)$ . Moreover, we have that the tubular neighborhood  $U_x^\delta$  of  $x$  defined by  $U_x^\delta := \{X^t(D_x) : t \in (-\delta, \delta)\}$ , where  $\delta > 0$  and  $D_x$  is a ball in the normal section to  $X(x)$  centered in  $x$ , is such that  $\hat{\mu}(U_x^\delta) > 0$ . Observe that, in a neighborhood of  $x$ , we can decompose the measure  $\hat{\mu}$  into  $\hat{\mu} = \mu \times \text{Leb}$  where  $\mu_\Sigma$  denote the measure induced by  $\hat{\mu}$  on  $\Sigma$  by the projection  $\pi$  along the flow direction determined by the tubular flow neighborhood and  $\text{Leb}$  is the length. Now, Poincaré recurrence assures that  $\hat{\mu}$ -a.e.  $y \in U_x^\delta$ , or equivalently  $\mu$ -a.e.  $z \in D_x$ , has infinitely many returns.  $\square$

Taking into account the previous result we can define by means of the tubular neighborhood theorem a time  $T > 0$  and a smooth function  $t : \Sigma_0 \rightarrow (-\delta, \delta)$  such that the Poincaré first return map

$$f_\Sigma : \Sigma_0 \rightarrow \Sigma$$

is well defined by  $f_\Sigma(y) = X^{T+t(y)}(y)$  in some open neighborhood  $\Sigma_0 \subset \Sigma$  of  $x$  in the section. If  $\Lambda \subset \mathcal{H}(K, \tau)$  is a subset of a hyperbolic block, then one can define the foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$  as in §2.2 for all  $y, z \in \mathcal{H}(K, \tau) \cap U_x^\delta$  and the intersection  $[y, z]_{\Sigma_x} := \mathcal{F}_y^u \cap \mathcal{F}_z^s$  consists of a unique point, provided that  $\delta$  is small. Moreover, by construction the foliations are invariant by the Poincaré map  $f_\Sigma$ .

Let  $\mu_\Sigma$  be as in Lemma 6.1. The measure  $\mu_\Sigma$  is clearly invariant by  $f_\Sigma$ . Then, by construction, not only  $x \in \text{supp}(\mu_\Sigma \mid_{\Lambda})$  as by the local product structure we have that  $\mu_\Sigma$  is equivalent to the product measure  $\mu_x^u \times \mu_x^s$ , where  $\mu_x^i$  are the conditional measures on  $\mathcal{F}_x^i$ . Moreover,

**Proposition 6.2.** *Consider a positive  $\hat{\mu}$ -measure set  $\Lambda \subset \mathcal{H}(K, \tau)$ . Given a regular point  $x \in \text{supp}(\hat{\mu} \mid_{\Lambda})$  there exists a smooth local cross section  $\Sigma$  and positive constants  $K', \tau'$ , such that for  $\mu_\Sigma$ -almost every  $z \in \Sigma$  we have*

$$d(f_\Sigma^n(y), f_\Sigma^n(z)) \leq K' e^{-\tau' n} d(y, z)$$

for all  $n \geq 1$  and every  $y, z$  in  $\mathcal{F}_z^s$ . Similar statement holds for  $\mathcal{F}_z^u$  with respect to  $f_\Sigma^{-1}$ .

*Proof.* This is a direct consequence of Lemma 2.3 and the subsequent paragraph.  $\square$

**6.2. Fiber-bunching for cocycles over Poincaré return maps.** Until the remaining of this section let  $f_\Sigma$  be a Poincaré return map that has a “good” hyperbolic structure on the foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$ . We consider the discrete-time induced cocycle over  $(f_\Sigma, \mu_\Sigma)$  for  $x \in \Sigma_0$  as

$$\Psi_H(x) = \Phi_H^{T+t(x)}(x). \quad (6.1)$$

It is not hard to check as before that the cocycle  $\Psi_H$  is Lipschitz continuous. Moreover, if  $(\Phi_H^t, \hat{\mu})$  has only zero Lyapunov exponents then the same property holds also for the cocycle  $(\Psi_H, \mu_\Sigma)$ . In fact, we have the following lemma.

**Lemma 6.3.** *Assume that  $\lambda^+(H, \mu) = 0$ , that is,  $\Phi_H^t$  has only zero Lyapunov exponents. Then for every  $\varepsilon > 0$  there exists  $T, \theta$  such that  $\mu(\mathcal{D}_H(T, \theta)) > 1 - \varepsilon$ .*

*Proof.* Since it is analogous to Corollary 2.4 in [38] we leave the details to the reader.  $\square$

We proceed to build stable and unstable holonomies for points in the same leaf of the foliations. Given  $H \in C^{r,\nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$ ,  $N > 0$  and  $\theta > 0$ , consider the set  $\mathcal{D}_H(N, \theta)$  of points  $x \in M$  satisfying

$$\prod_{j=0}^{k-1} \|\Psi_H(f_\Sigma^{jN}(x))\| \|\Psi_H(f_\Sigma^{jN}(x))^{-1}\| \leq e^{kN\theta} \quad \text{for all } k \geq 1 \quad (6.2)$$

and the dual relation for  $\Phi_H^{-T}$  with relation to  $X^{-T}$ . Let  $\mathcal{O}$  be a *holonomy block* for  $H$  for  $f_\Sigma$  if it is a compact subset of  $\mathcal{H}(K, \tau) \cap \mathcal{D}_H(N, \theta)$  for some constants  $K, \tau, T, \theta$  satisfying  $3\theta < \tau$ . Observe that domination is an open condition for the cocycle and this enables to obtain strong-stable and strong-unstable foliations for all nearby cocycles. More precisely, using that the foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$  inherit the hyperbolicity from the hyperbolic block (possibly with some larger constants) we prove the existence of holonomies similarly to the discrete time setting. In fact, the same ideas as in Proposition 4.2 yields that:

**Proposition 6.4.** *For every  $x \in \mathcal{O}$  and  $y, z \in \mathcal{F}_x^u$ , there exists  $C_2 > 0$  and a symplectic linear transformation  $L_{H,x,y}^u : \{y\} \times P\mathbb{K}^{2\ell} \rightarrow \{z\} \times P\mathbb{K}^{2\ell}$  such that:*

- (1)  $L_{x,x}^u = \text{id}$  and  $L_{x,z}^u = L_{y,z}^u \circ L_{x,y}^u$
- (2)  $\Psi_H(f_\Sigma^{-1}(z)) \circ L_{H,f_\Sigma^{-1}(y),f_\Sigma^{-1}(z)}^u \circ \Phi_H^t(y)^{-1} = L_{H,y,z}^u$  for all  $t \geq 0$  and
- (3)  $\|L_{H,y,z}^u - \text{id}\| \leq C_2 d(y, z)$ .

As before, some consequences are the continuous disintegration of invariant measures. Given an holonomy block  $\mathcal{O}$  and a regular point  $x$  let  $N_x^s(\mathcal{O}, \delta) \subset N_x^s(\delta)$  the subset of  $\Sigma_x$  obtained by replacing  $\mathcal{H}(K, \tau)$  by the holonomy block  $\mathcal{O}$  and define  $N_x^u(\mathcal{O}, \delta)$  and  $N_x(\mathcal{O}, \delta)$  analogously. Let  $\psi_H$  denote the projectivized version of the cocycle  $\Psi_H$  and, by some abuse of notation, consider  $h_{x,y}^u$  the projectivized holonomy  $L_{H,x,y}^u$ . The next proposition asserts that one can obtain a continuous disintegration of invariant probabilities that project to  $\mu_\Sigma$ .

**Proposition 6.5.** *Let  $\mathcal{O}$  be a positive  $\mu$ -measure holonomy block, consider  $x \in \text{supp}(\mu_\Sigma \mid \mathcal{O})$  and set the neighborhoods  $N_x^s(\mathcal{O}, \delta)$ ,  $N_x^u(\mathcal{O}, \delta)$  and  $N_x(\mathcal{O}, \delta)$  as above. Then every  $\psi_H^t$ -invariant probability measure  $m$  with  $\Pi_* m = \mu_\Sigma$  admits a continuous disintegration on  $\text{supp}(\mu_\Sigma \mid N_x(\mathcal{O}, \delta))$ . Moreover,*

$$m_z = (h_{y,z}^s)_* m_y \quad \text{and} \quad m_z = (h_{w,z}^u)_* m_y$$

*for all  $y, z, w \in \text{supp}(\mu \mid N_x(\mathcal{O}, \delta))$  such that  $y, z$  belong to the same  $\mathcal{F}^s$  leaf and  $z, w$  belong to the same  $\mathcal{F}^u$  leaf.*

*Proof.* This proof is analogous to Proposition 4.3 and uses the nonuniform hyperbolicity of the Poincaré first return map in the same way as in [38]. For that reason we shall omit the details and leave the proof as a thorough exercise to the reader, nonetheless is accomplished by borrowing the arguments in [38, Proposition 3.1].  $\square$

**6.3. A lot of closed orbits inside holonomy blocks.** In order to go on with the proof of our results we also need to use the continuous-time version of [38, Proposition 4.5], included in (1) and (2) of Proposition 6.7 below, and which is supported in Katok's shadowing lemma for non-uniformly hyperbolic systems proved in [25]. We observe that the flows version of the Katok theorem was treated recently in a more general context by Lian and Young (cf. [37, §1.2]). But before we state it we shall introduce some elementary notation typical of Pesin's theory framework. Recall that the hyperbolicity (uniform and non-uniform) for flows is often defined with respect to the linear Poincaré flow (cf. [9]). From now on we consider a  $C^{1+\alpha}$  flow  $X^t: M \rightarrow M$  on a  $d$ -dimensional manifold  $M$ . Given  $k = 0, \dots, d-1$ ,  $\ell > 1$  and  $\chi > 0$  we denote by  $\Lambda_{\chi, \ell}^k$  the set of points  $x \in M$  defining a *Pesin hyperbolic block* cf. [25, §2] but with respect to the decomposition of the normal bundle at  $x$ ,  $N_x = N_x^s \oplus N_x^u$ , where  $\dim(N_x^s) = k$ . The set  $\Lambda_j^k := \Lambda_{\chi_j, \ell_j}^k$  is the hyperbolic block with index  $k$ , Lyapunov exponent  $\chi_j$ , and constant  $\ell_j$ , where  $\mu(\cup_i \Lambda_j^i)$  tends to 1, and  $\chi_j$  and  $\ell_j$  tend to  $\infty$  as  $j \rightarrow +\infty$ . Since we assume an ergodic base flow we have a constant index  $k$  and thus omit it from now on, that is  $\Lambda_{\chi, \ell} = \mathcal{K}(\ell, \chi)$  using the notation in §2.1.

We are in conditions to present the statement of [25, Main Lemma pp. 154] but for the flow context.

**Theorem 6.6.** (*Katok's shadowing lemma for flows*) *Fixed any  $j \geq 1$  (thus  $\chi_j > 0$  and  $\ell_j > 1$ ), there exist positive numbers  $K, \tau, \rho$  and  $T$ , such that given  $\delta > 0$ , there exists  $\varepsilon = \varepsilon(d, j, \delta) > 0$  where the following holds: if for a given  $z \in \Lambda_j$  and  $\hat{\pi} > 0$  we have  $X^{\hat{\pi}}(z) \in \Lambda_j$  and also  $d(z, X^{\hat{\pi}}(z)) < \varepsilon$ , then there exists  $p = p(z) \in M$  such that:*

- (1) (*closing*)  $p$  is closed of period  $\pi \in (\hat{\pi} - T, \hat{\pi} + T)$ , i.e.  $X^\pi(p) = p$ ;
- (2) (*shadowing*)  $d(X^t(p), X^t(z)) < \delta$  for all  $t \in [0, \pi]$ ;
- (3) (*hyperbolicity*)  $p$  is hyperbolic for the linear Poincaré flow  $P_X^\pi(p)$ ;
- (4) (*uniform hyperbolicity*) the eigenvalues  $\alpha$  of  $P_X^\pi(p)$  satisfy  $|\log |\alpha|| > \pi\tau$ ;
- (5) (*stable manifold*) for all  $t > 0$  and  $x, y \in W_{loc}^s(p)$  we have  $d(X^t(x), X^t(y)) < Ke^{-\tau t}d(x, y)$ ;
- (6) (*unstable manifold*) for all  $t > 0$  and  $x, y \in W_{loc}^u(p)$  we have  $d(X^{-t}(x), X^{-t}(y)) < Ke^{-\tau t}d(x, y)$ ;
- (7) (*uniform size*) both  $W_{loc}^s(p)$  and  $W_{loc}^u(p)$  have size larger than  $\rho$  and
- (8) (*transversality*) for all points  $w \in \Lambda_j$  in a  $\rho$ -neighborhood of  $z$ , there exist small  $t_w, s_w \in \mathbb{R}$ , such that we have that  $W_{loc}^s(p)$  intersects  $W_{loc}^u(X^{t_w}(w))$  at exactly one point and  $W_{loc}^u(p)$  intersects  $W_{loc}^s(X^{s_w}(w))$  at exactly one point.

Next result is the continuous-time version of Proposition 4.8. Let  $M_0 := \{x \in M: \lambda^+(H, \mu) = 0\}$  so let us assume that  $\mu(M_0) > 0$ .

**Proposition 6.7.** *Given  $\hat{\varepsilon} > 0$  and  $k \geq 2$  there exists a holonomy block  $\tilde{O}$  for  $H$  so that  $\mu(M_0 \setminus \tilde{O}) < \hat{\varepsilon}$ , distinct dominated periodic points  $\{p_i\}_{i=1}^k$  in  $\tilde{O}$  and a Hamiltonian linear differential system  $\tilde{H} \in C^{r, \nu}(M, \mathfrak{sp}(2\ell, \mathbb{K}))$  such that the following properties hold:*

- (1)  $W_{loc}^u(p_i) \cap W_{loc}^s(p_{i+1}) \neq \emptyset$  consists of one point for all  $1 \leq i \leq k$ ;
- (2)  $p_i \in \text{supp}(\mu \mid \tilde{O} \cap X^{-\pi_i}(\tilde{O}))$ , where  $\pi_i$  denotes the period of  $p_i$ ;
- (3)  $\|A - B\|_{r, \nu} < \hat{\varepsilon}$ ;



(4) the Lyapunov spectrum of  $B^{\pi_i}(p_i)$  is real and simple.

Finally, the set of cocycles  $B$  satisfying (1), (2) and (4) is open in the  $C^{r,\nu}$ -topology.

*Proof.* The strategy to obtain (1) and (2) is modeled in [38, Proposition 4.5] and strongly uses Theorem 6.6. We recall the highlights of Viana's proof borrowing the arguments in [38, §4.2].

Step 1: Given  $j$  (i.e.  $\chi_j$  and  $\ell_j$ ) such that  $\mu(M_0 \setminus \Lambda_j) < \hat{\varepsilon}/2$ , we fix  $K, \tau, \rho$  and  $T$  as in Theorem 6.6. Let  $\theta > 0$  be such that  $3\theta < \tau$ . The Lemma 6.3 assures that for  $\mu$ -a.e.  $x \in M_0$  there exists  $T > 0$  such that  $x \in \mathcal{D}_H(T, \theta)$ . Choose  $T$  large enough so that  $\mu(M_0 \setminus \mathcal{D}_H(T, \theta)) < \hat{\varepsilon}/2$ . We take a holonomy block defined by  $\mathcal{O} = \Lambda_j \cap \mathcal{D}_H(T, \theta)$  such that  $\mu(M_0 \setminus \mathcal{O}) < \hat{\varepsilon}$  and  $\mu(\mathcal{O}) > 0$ ;

Step 2: Fixing  $\varepsilon > 0$ , we find  $k$  distinct points  $\{z_i\}_{i=1}^k \subset M$  and  $\{\pi_i\}_{i=1}^k \subset \mathbb{R}$  such that  $z_i$  and  $X^{\pi_i}(z_i)$  are in  $B(x, \rho/2)$ ,  $d(z_i, X^{\pi_i}(z_i)) < \varepsilon$  and  $z_i \in \text{supp}(\mu|_{\mathcal{O} \cap X^{-\pi_i}(\mathcal{O})})$ . We may assume also that all  $z_i$ 's are at a distance larger than a fixed  $r > 0$ . Now, we are in condition to apply Theorem 6.6 and complete part (1) of the lemma;

Step 3: Given  $\delta = r/2$ , there exists  $\varepsilon = \varepsilon(d, j, \delta) > 0$  given by Theorem 6.6 such that when feeding Step 2 with this  $\varepsilon$  the following holds: if for a given  $z_i \in \mathcal{O}$  and  $\hat{\pi}_i > 0$  we have  $X^{\hat{\pi}_i}(z_i) \in \mathcal{O}$  and also  $d(z_i, X^{\hat{\pi}_i}(z_i)) < \varepsilon$ , then there exists distinct  $p_i = p_i(z_i) \in M$  such that:

- (a)  $p_i$  is closed of period  $\pi_i \in (\hat{\pi}_i - T, \hat{\pi}_i + T)$ , i.e.  $X^{\pi_i}(p_i) = p_i$ ;
- (b)  $d(X^t(p_i), X^t(z_i)) < \delta$  for all  $t \in [0, \pi_i]$ ;
- (c)  $p_i$  is hyperbolic for the linear Poincaré flow  $P_X^{\pi_i}(p_i)$ ;
- (d) the eigenvalues  $\alpha$  of  $P_X^{\pi_i}(p_i)$  satisfy  $|\log |\alpha|| > \pi_i \tau$ ;
- (e) for all  $t > 0$  and  $x, y \in W_{loc}^s(p_i)$  we have  $d(X^t(x), X^t(y)) < Ke^{-\tau t}d(x, y)$ ;
- (f) for all  $t > 0$  and  $x, y \in W_{loc}^u(p_i)$  we have  $d(X^{-t}(x), X^{-t}(y)) < Ke^{-\tau t}d(x, y)$ ;
- (g) both  $W_{loc}^s(p_i)$  and  $W_{loc}^u(p_i)$  have size larger than  $\rho$  and
- (h) for all points  $w \in \mathcal{O}$  in a  $\rho$ -neighborhood of  $z_i$  there exist small  $t_w, s_w \in \mathbb{R}$ , such that we have that  $W_{loc}^s(p_i)$  intersects  $W_{loc}^u(X^{t_w}(w))$  at exactly one point and  $W_{loc}^u(p_i)$  intersects  $W_{loc}^s(X^{s_w}(w))$  at exactly one point.

Step 4: Now we will prove part (2) of the lemma. We start to define the subset  $\tilde{\mathcal{O}}$ . By Step 2 we have  $z_i \in \text{supp}(\mu|_{\mathcal{O} \cap X^{-\pi_i}(\mathcal{O})})$ , so define a compact set  $\mathcal{O}_i$  such that  $\mathcal{O}_i \subset B(z_i, \nu) \cap \mathcal{O}$  and  $X^{\pi_i}(\mathcal{O}_i) \subset B(X^{\pi_i}(z_i), \nu) \cap \mathcal{O}$  for some very small  $\nu > 0$ . Using the transversality given in (h) of Step 3 we obtain that, for any  $w \in \mathcal{O}_i$ , there exist  $t_w, s_w$  such that  $W_{loc}^s(p_i)$  intersects  $W_{loc}^u(X^{t_w}(w))$  at exactly one point and  $W_{loc}^u(p_i)$  intersects  $W_{loc}^s(X^{s_w}(w))$ . Let  $\Gamma_i^s \subset W_{loc}^s(p_i)$ , respectively  $\Gamma_i^u \subset W_{loc}^u(p_i)$ , stand for those intersections. Now, for all  $k, l \in \mathbb{N}$ , we define  $\Gamma_i^u(k) = X^{-\pi_i k}(\Gamma_i^u)$  and  $\Gamma_i^s(l) = X^{\pi_i l}(\Gamma_i^s)$ . A standard  $\lambda$ -lemma argument assures that, for any  $k, l$ , the local stable manifolds of points in  $\Gamma_i^u(k)$  intersects in a transversal uniform way the local unstable manifold set of points in  $\Gamma_i^s(l)$ . Let  $\mathcal{O}_i(k, l)$  denote that intersection. We fatten  $\mathcal{O}_i(k, l)$  by a time  $t$  where  $t := \frac{1}{2} \min\{t_w, s_w\}$  and  $w \in \mathcal{O}_i$  (for all  $i$ ) and we let  $\hat{\mathcal{O}}_i(k, l) := \cup_{s \in [-t, t]} X^s(\mathcal{O}_i(k, l))$ . Finally, we define the set  $\tilde{\mathcal{O}}$  by:

$$\tilde{\mathcal{O}} := \mathcal{O} \bigcup_{k+l \geq 1} \hat{\mathcal{O}}_i(k, l).$$

Observe that  $\mu(M_0 \setminus \tilde{O}) < \varepsilon$ . Figure 1 of [38] is a nice illustration of what is happening inside a Poincaré section of the closed orbit  $p_i$ .

- Step 4: We obtain that  $\tilde{O}$  displays uniform hyperbolic rates. More precisely, that there exists  $K' > K$  such that points  $x, y$  in the local invariant manifolds of any  $\xi \in \tilde{O}$  satisfy the inequalities:  $d(X^t(x), X^t(y)) < K'e^{-\tau t}d(x, y)$  and  $d(X^{-t}(x), X^{-t}(y)) < K'e^{-\tau t}d(x, y)$ . The key ingredient is the continuity of the invariant manifolds (see [38, Lemma 4.9]) and the stability, on small segments of orbits, of the local product structure.
- Step 5: Now we show that  $\tilde{O}$  is still a holonomy block. That is, there exists  $\theta' > \theta$  (but such that  $3\theta' < \tau$ ) such that  $\tilde{O} \subset \mathcal{D}_H(T, \theta')$ . The arguments are like the ones in [38, Lemma 4.10] and we leave the details to the reader.
- Step 6: Finally, we just have to prove that  $p_i \in \text{supp}(\mu \mid \tilde{O} \cap X^{-\pi_i}(\tilde{O}))$ . We observe that for any  $k, l > 0$  we have

$$X^{\pi_i}(O_i(k, l-1)) = O_i(k-1, l). \quad (6.3)$$

We claim that, for all  $k + l \geq 1$ ,  $\mu(\hat{O}_i(k, l)) > 0$ . It is sufficient to show that  $\mu^u \times \mu^s(O_i(k, l)) > 0$  where this measure was treated in Definition 2.4 and this can be achieved by borrowing the arguments in [38, Lemma 4.11]. We get that  $p_i$  is accumulated by sets  $O_i(k, l)$ , thus by sets  $\hat{O}_i(k, l)$ , and which are inside  $\tilde{O}$ . Using (6.3) we get that the sets  $O_i(k, l)$  are also inside  $X^{-\pi_i}(O \cup_{k+l \geq 1} O_i(k, l))$ . Therefore,  $p_i \in \text{supp}(\mu \mid \tilde{O} \cap X^{-\pi_i}(\tilde{O}))$  and (2) is proved.

- Step 7: In order to obtain (3) and (4) we proceed as in the proof of Proposition 4.8 but using the Lemma 5.6 in §5.3 which is the Hamiltonian perturbation tool which allows us to perform the continuous-time perturbation in the vein of the one in the proof of Proposition 4.8.

□

**6.4. Proof of Theorem B for general flows.** The strategy follows the same steps as the ones described in §4 (discrete-time case) and also in §5 (continuous-time case with suspension flow in the base). There are essentially three main novelty key points:

- obtaining the closed orbits which was performed in §6.3;
- the reduction of the study of hyperbolicity on the normal cross sections cf. §6.1 and
- the using of the induced return cocycle described in §6.2.

We begin by using the construction developed in §6.3 in order to obtain a large quantity of closed orbits near  $x \in \text{supp}(\mu \mid O)$  where  $O$  is a positive  $\mu$ -measure holonomy block. Of course that those closed orbits can be seen as closed orbits associated to the Poincaré map  $\mathcal{P}'_x$  in a cross section  $\Sigma$  and very near from  $x$ .

Then, since those closed orbits are hyperbolic we have large leaves  $\mathcal{F}^u$  and  $\mathcal{F}^s$ . Hence, we can use the  $\lambda$ -lemma and build horseshoes and thus, a symbolic dynamics obtaining closed orbits with very large period which can be turn, via the perturbation Lemma 5.6, into closed orbits with real and simple spectrum cf. Proposition 6.7 (see also the final part of the proof in Proposition 4.8).

Finally, the usual type of perturbation is done to break the holonomy. We use the definition of domination (in (6.2)) with respect to the induced cocycle  $\Psi_H$  defined in (6.1). Moreover, we

act with the holonomies along the foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$  cf. Proposition 6.4. The perturbation is carried out using Lemma 5.6 and intend to spoil the action of the induced cocycle  $\Psi_H$  defined in (6.1) and it is quite similar to the one performed in the Section 5. We observe that the perturbation of the Hamiltonian linear differential system is done in a small flowbox tubular neighborhood  $\mathcal{T}$  of a given heteroclinic point  $z$ , forward asymptotic with the closed orbit  $p$  and backward asymptotic with the closed orbit  $q$  (recall §5.4). For this reason our perturbation of the stable holonomy cannot interfere with the unstable holonomy which remain with the same action because  $\cup_{t>0} X^t(\mathcal{T})$  is far from the backward iterates of  $\mathcal{T}$ .

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